Incremental Computation of the Homology of Generalized Maps: An Application of Effective Homology Results

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September 30, 2015

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Abstract

This paper deals with the incremental computation of the homology of "cellular" combinatorial structures, namely combinatorial maps and incidence graphs. "Incremental" is related to the operations which are applied to construct such structures: basic operations, i.e. the creation of cells and the identification of cells, are considered in the paper. Such incremental computation is done by applying results of *effective homology* [RS06]: a correspondence between the chain complex associated with a given combinatorial structure is maintained with a "smaller" chain complex, from which the homology groups and homology generators can be more efficiently computed.

1 Introduction

Many works deal with the computation of the homology of subdivided geometric objects. Usually, one intend to compute the homological information of a given object, i.e. its homology groups and/or homology generators [EH10], and many methods have been conceived and optimized, for instance based on Smith Normal Form (e.g. [Sto96, Gie96, DSV01, DHSW03, PAFL06]), or on simplifications as elementary reductions (e.g. [MB09, GDJMR09, DKMW11]).

This paper deals with the *incremental* computation of the homological information during a construction process: given an initial object I, its homological information H_I, and an operation O applied to I producing the resulting object R, how is it possible to efficiently deduce the homological information H_R of R from H_I and O ?

For some operations, *effective homology* results [RS06] make it possible to incrementally compute the homological information. These results stand for any chain complex, and their application for (chain complexes associated with) *simplicial* combinatorial structures, as simplicial sets [May67] and subclasses of abstract simplicial complexes [Ago76], has been studied in [BMLH10, BCMA$^+$11] for instance.

The main tools and results on which this paper is based are:

- the notion of *homological equivalence*: it makes it possible to maintain an association between a chain complex C_* and a "small" chain complex C_*^S, which are homologically equivalent; so, the homological information of C^* can be efficiently deduced from C_*^S;
- the *perturbation lemmas*, which make it possible to modify the boundary operators within a homological equivalence, under some assumptions;
- the *Short Exact Sequence Theorem*, or SES theorem, which makes it possible to incrementally compute the homological information of a chain complex for some operations.

These notions and results are here applied for *cellular* combinatorial structures, as incidence graphs and combinatorial maps [DL14]. Such combinatorial structures have been defined and studied in order to represent the structure of subdivided geometric objects, for instance for applications in Geometric Modeling, Computational Geometry, Image Processing and Analysis, etc. For many applications, the cells of an object are not necessarily convex; moreover, it is not possible to combinatorially decide whether the geometric realization of a cell is homeomorphic or not to a ball; so, cells of such cellular combinatorial structures exist, which geometric realizations are not homeomorphic to balls. It is thus not possible to directly apply to such structures many results about homology.

As far as we know, a *simplicial analog* can be associated with any combinatorial cellular structure [Bri93, Lie94, Ber99]. So, it is possible to compute the homology of a cellular structure by computing the homology of its simplicial analog, but the optimization of the cellular representation is lost. In other words, there are many more simplices in a simplicial analog than cells in the cellular object. The methods presented in [Bas10] and [ADLP15, ALP15] optimize the computation of the homological information by taking into account the notion of cell, but it is restricted to a subclass of cellular structures such that the cells satisfy some combinatorial constraints (informally speaking, the homology of a cell has to be equivalent to the homology of a ball). Moreover, these methods are not incremental ones.

The incremental method presented in this paper takes also the notion of cell into account, and it can be applied to a wider class of cellular structures. The key idea consists in handling the homological information of the object together with the homological information of each of its cells.

More precisely, the incremental computation of the homology of generalized maps is first studied, and then extended for other cellular structures. A generalized map, or gmap, corresponds to a cellular quasi-manifold (cf. Section 4.1). It can be constructed by applying two basic operations: extension (or cellular cone), and sewing (or identification of two isomorphic cells, together with their boundaries)[1].

The main results presented in the paper are the following. A homological equivalence can be associated with any gmap and any of its cells, and:

- it is possible to compute a homological equivalence for the gmap (and its cells) resulting from an extension, in linear time according to the initial homological equivalences, by applying the perturbation lemmas (cf. theorem 4.10);
- it is possible to compute a homological equivalence for the gmap (and its cells) resulting from a sewing, by one application of the SES theorem (cf. theorem 4.11). This result is available for gmaps without self-bending (cf. Section 4.1). The complexity of the computation is also studied, which depends also on the operations previously applied in the construction process;
- these results can be extended for other cellular structures, as chains of maps for the representation of "cellular" complexes, n-surfaces which are quasi-manifolds in which cells are not incident several times to other cells, and incidence graphs.

In order to introduce the study for cellular structures, the same approach is first followed for semi-simplicial sets. Lemma 3.6 and theorem 3.7 are also a result of this work, showing that the identification operation for simplices can be expressed as a short exact sequence. Note that theorem 3.7 generalizes the *Mayer-Vietoris algorithm* presented in [BMLH10].

The outline of this paper is:

- main notions and results of effective homology are recalled in Section 2;
- the definitions of semi-simplicial sets, cone and identification operations, are recalled in Section 3.1. The incremental computation of the homological information for the cone and identification operations is studied in Section 3.2;
- the definitions of generalized maps, extension and sewing operations, and simplicial analogs are recalled in Section 4.1. The incremental computation of the homological information for the extension and sewing operations is studied in Section 4.2. The complexity and the extensions for other cellular structures are also discussed;
- we conclude in Section 5.

[1] As a simplicial structure can be constructed by the cone operation, for creating the simplices, and the identification of simplices, for "gluing" them together.

2 Effective homology bases

This section is mainly based on the course notes of J. Rubio and F. Sergeraert [RS06]. Some of their results are also presented in [BMLH10].

Definition 2.1 *Chain complex [RS06]*
A chain complex (C, ∂) *is defined by* C*, a graded module*[2] *of type* $\mathbb{Z}$ *over the commutative graded ring*[3] $\mathbb{Z}$*, and* $\partial : C \mapsto C$*, a boundary operator, i.e. a graded module morphism of degree* -1 *such that*[4] *the composition* $\partial\partial = 0$*.*

For any q in $\mathbb{Z}$, C_q is the group of *q-chains*. The image $B_q = C_{q+1}\partial_{q+1} \subset C_q$ is the (sub)group of *q-boundaries*. The kernel $Z_q = Ker(\partial_q) \subset C_q$ is the group of *q-cycles*. Since $\partial_{q+1}\partial_q = 0$, $B_q \subset Z_q$; $H_q = Z_q/B_q$ is the q^{th} *homology group*[5] of C.

Definition 2.2 *Chain complex morphism*
A chain complex morphism Φ *between two chain complexes* (C^0, ∂^0) *and* (C^1, ∂^1) *is a graded module morphism between* C^0 *and* C^1 *such that* $c\partial^0\Phi = c\Phi\partial^1$*, for any chain* $c \in C^0$*.*

Definition 2.3 *Homology*
The homology H *of* (C, ∂) *is the direct sum of its homology groups. By extension, the chain complex* $(H, 0)$ *is also called the homology of* (C, ∂)*.*

2.1 Effective homology tools

2.1.1 Reduction

A reduction associates with some chain complex a generally "smaller" one [GDJMR09, KMM04, RS06].

Definition 2.4 *Reduction*
A reduction $\rho : (C^0, \partial^0) \ggg (C^1, \partial^1)$ *is a diagram:*

$$\rho = \boxed{\overset{h}{\circlearrowright} (C^0, \partial^0) \underset{f}{\overset{g}{\leftrightarrows}} (C^1, \partial^1)}$$

where:

- (C^0, ∂^0) *and* (C^1, ∂^1) *are chain-complexes*
- f *and* g *are chain-complex morphisms*

[2] See [Bou89] for the definitions of basic algebraic notions, and more precisely (II.11.2 p366) for the definition of graded modules. These notions are recalled in Annex 6.1 page 48.
[3] The graduation is the trivial graduation, i.e. $\mathbb{Z}_0 = \mathbb{Z}$, and $\mathbb{Z}_i = 0$ for $i \neq 0$.
[4] Note that $x\delta$ denotes $\delta(x)$ and $x\delta\epsilon$ denotes $\epsilon(\delta(x))$.
[5] In fact, all these groups: chain, boundary, cycle, homology, are graded modules.

- *h is a* homotopy operator*, i.e. a graded module morphism of degree* $+1$

which satisfy the relations:

(a) $gf = id_{C^1}$

(b) $fg + h\partial^0 + \partial^0 h = id_{C^0}$

(c) $hf = gh = hh = 0$

ρ is also denoted by $((C^0, \partial^0), (C^1, \partial^1), h, f, g)$, and will be sometimes symbolized by: $(C^0, \partial^0) \overset{\rho}{\ggg} (C^1, \partial^1)$.

Several reductions are described in Example 7 and Example 8 in Annex 6.2 pages 52 and 55, on which the definition properties *a), b) and c)* are explained.

Property 2.5 *If a reduction exists between two chain complexes (C^0, ∂^0) and (C^1, ∂^1), then their homologies are isomorphic.*

Definition 2.6 *Composition of reductions*
Let $\rho^{01} = ((C^0, \partial^0), (C^1, \partial^1), h^{01}, f^{01}, g^{01})$ and $\rho^{12} = ((C^1, \partial^1), (C^2, \partial^2), h^{12}, f^{12}, g^{12})$ be two reductions.
The composition $\rho^{01}\rho^{12}$ is $((C^0, \partial^0), (C^2, \partial^2), h^{02}, f^{02}, g^{02})$ where:

- $h^{02} = h^{01} + f^{01} h^{12} g^{01}$
- $f^{02} = f^{01} f^{12}$
- $g^{02} = g^{12} g^{01}$

Property 2.7 *The composition of two reductions is a reduction.*

2.1.2 Homological equivalence

Definition 2.8 *A homological equivalence Υ between two chain-complexes, (C, ∂) and (C^S, ∂^S) is a pair of reductions ρ and ρ^S involving a chain complex (C^B, ∂^B), such that:*

$$\Upsilon : (C, \partial) \overset{\rho}{\lll} (C^B, \partial^B) \overset{\rho^S}{\ggg} (C^S, \partial^S)$$

It is also denoted by

$$\Upsilon : (C, \partial) \lll\!\!\ggg (C^S, \partial^S)$$

This notion allows to compare chain complexes which are not directly related by a reduction (see Figure 1). From a practical point of view, this notion is used to associate a "small" chain complex (C^S, ∂^S) with another chain complex (C, ∂), through a "bigger" one (C^B, ∂^B) (see Section 3). Note that the homologies of all these chains complexes are isomorphic, as a straightforward consequence of Definition 2.8 and Property 2.5. Hence, since (C^S, ∂^S) is "smaller" than (C, ∂), meaning C^S contains less generators than C, the homology of (C, ∂) can be efficiently computed by computing the homology of (C^S, ∂^S).

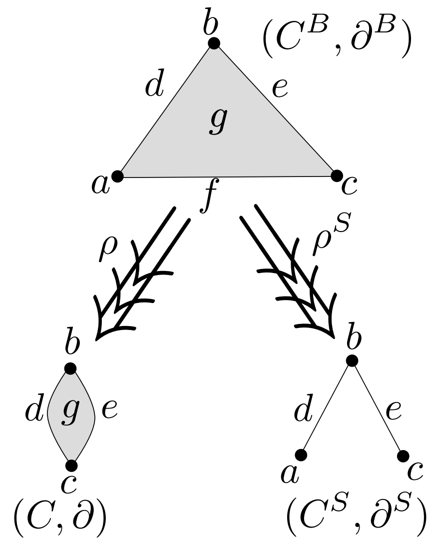


Figure 1: Homological equivalence between (C, ∂) and (C^S, ∂^s) through (C^B, ∂^B); reduction ρ (resp. ρ^S) is characterized by the fact that the image of a (resp. f) by the homotopy operator is f (resp. g).

2.1.3 Cone of a morphism

Definition 2.9 *Let ϕ be a chain complex morphism which maps (C^0, ∂^0) onto (C^1, ∂^1). The cone of ϕ, $Cone(\phi) = (C^\phi, \partial^\phi)$, is the chain complex defined by:*

$$\forall i, C_i^\phi = C_{i-1}^0 \oplus C_i^1$$

$$\text{and } \partial_i^\phi = \begin{pmatrix} -\partial_{i-1}^0 & \phi_{i-1} \\ 0 & \partial_i^1 \end{pmatrix}$$

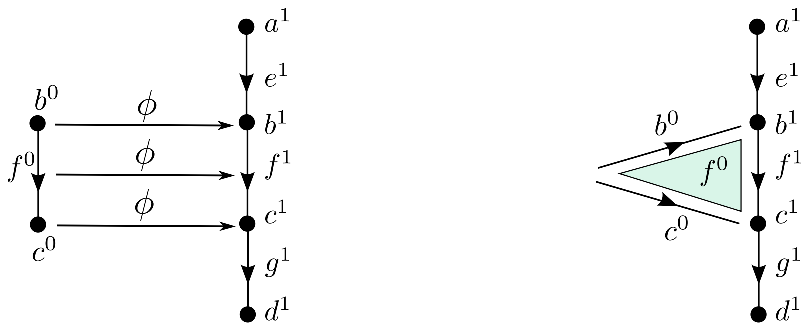
C_i^ϕ is the direct sum of C_{i-1}^0 and C_i^1, $\forall i$. Each element $c_i = c_{i-1}^0 + c_i^1$ of C_i^ϕ is denoted $c_i = (c_{i-1}^0 \quad c_i^1)$, where $c_{i-1}^0 \in C_{i-1}^0$ and $c_i^1 \in C_i^1$, and:

$$c_i \partial_i^\phi = (c_{i-1}^0 \quad c_i^1) \begin{pmatrix} -\partial_{i-1}^0 & \phi_{i-1} \\ 0 & \partial_i^1 \end{pmatrix} = (-c_{i-1}^0 \partial_{i-1}^0 \quad c_{i-1}^0 \phi_{i-1} + c_i^1 \partial_i^1)$$

The consistence of Definition 3.2 is recalled in Annex 6.3 page 56.

Example 1 *This example is illustrated on Figure 2. Let (C^0, ∂^0) and (C^1, ∂^1) be two chain complexes defined by:*

C^0		b^0	c^0			f^0	
∂^0		0	0			c^0-b^0	
C^1	a^1	b^1	c^1	d^1	e^1	f^1	g^1
∂^1	0	0	0	0	b^1-a^1	c^1-b^1	$d^1 - c^1$



(a) Graphical representation of two chain complexes (C^0, ∂^0) and (C^1, ∂^1), and a chain complex morphism ϕ between them

(b) Graphical representation of (C^ϕ, ∂^ϕ)

Figure 2: Example of the cone of a chain-complex morphism

Let ϕ be defined by: $b^0 \longrightarrow b^1$, $c^0 \longrightarrow c^1$, $f^0 \longrightarrow f^1$. Then, for instance:

$$(0 \quad e^1) \begin{pmatrix} -\partial_0^0 & \phi_0 \\ 0 & \partial_1^1 \end{pmatrix} = (0 \quad b^1 - a^1)$$

$$(b^0 \quad 0) \begin{pmatrix} -\partial_0^0 & \phi_0 \\ 0 & \partial_1^1 \end{pmatrix} = (0 \quad b^1)$$

$$(f^0 \quad 0) \begin{pmatrix} -\partial_1^0 & \phi_1 \\ 0 & \partial_2^1 \end{pmatrix} = (-c^0 + b^0 \quad f^1)$$

and (C^ϕ, ∂^ϕ) is defined by:

C^ϕ	a^1	b^1	c^1	d^1	e^1	f^1	g^1	b^0	c^0	f^0
∂^ϕ	0	0	0	0	b^1-a^1	c^1-b^1	$d^1 - c^1$	b^1	c^1	$b^0 - c^0 + f^1$

□

Note that the cone of a morphism is simply the representation of a chain complex morphism as a chain complex. It contains no more information.

2.1.4 Perturbation and related lemmas

Definition 2.10 *Perturbation ([RS06] page 48)*

Le (C, ∂) be a chain complex. Let $\delta : C \longrightarrow C$ be a graded module morphism of degree -1, such that $(\partial + \delta)^2 = 0$. δ is called a perturbation of ∂.

Example: let (C, ∂) be a chain complex; since $(C, 0)$ also is a chain complex, ∂ is a perturbation of the null boundary operator.

Property 2.11 *Easy perturbation lemma ([RS06] page 48-49)*
Let $\rho = ((C^0, \partial^0), (C^1, \partial^1), h, f, g)$ *be a reduction, and let* δ^1 *be a perturbation of* ∂^1*, then* $\rho^\delta = ((C^0, \partial^0 + \delta^0), (C^1, \partial^1 + \delta^1), h, f, g)$ *is a reduction, where* $\delta^0 = f\delta^1 g$.

Property 2.12 *Basic perturbation lemma ([RS06] page 49)*
Let $\rho = ((C^0, \partial^0), (C^1, \partial^1), h, f, g)$ *be a reduction, and let* δ^0 *be a perturbation of* ∂^0 *which satisfies the nilpotency hypothesis, i.e.:*

$$\forall c \in C^0, \exists i \in \mathbb{N} \quad | \quad c(\delta^0 h)^i = 0$$

Then $\rho^\delta = ((C^0, \partial^0 + \delta^0), (C^1, \partial^1 + \delta^1), h^\delta, f^\delta, g^\delta)$ *is a reduction, where:*

- $\delta^1 = g\Phi\delta^0 f = g\delta^0\Psi f$.
- $f^\delta = \Psi f$
- $g^\delta = g\Phi$
- $h^\delta = h\Phi = \Psi h$

with[6] $\Phi = \sum_{i=0}^{i=\infty}(-1)^i(\delta^0 h)^i$ *and* $\Psi = \sum_{i=0}^{i=\infty}(-1)^i(h\delta^0)^i$.

The proofs of the Easy and Basic Perturbation Lemmas are recalled in Annex 6.4 page 56.

2.2 SES theorem

Definition 2.13 *Short Exact Sequence ([RS06] page 71)*
An effective short exact sequence of chain complexes is a diagram:

$$0 \xrightarrow{0} (C^0, \partial^0) \underset{i}{\overset{r}{\leftrightarrows}} (C^1, \partial^1) \underset{j}{\overset{s}{\leftrightarrows}} (C^2, \partial^2) \xrightarrow{0} 0$$

where

- (C^0, ∂^0), (C^1, ∂^1), (C^2, ∂^2) *are chain complexes,*
- *i,j are chain complex morphisms,*
- *r,s are graded module morphisms,*

which satisfy

1. $ir = id_{C^0}$

[6] Due to the nilpotency hypothesis, each sum contains a finite number of non null terms.

2. $ri + js = id_{C^1}$

3. $sj = id_{C^2}$

Note that i,s are injective, and j, r are surjective due to properties 1 and 3. Several properties can also be deduced. For instance, $ij = 0$ (since r is surjective and $rij = (id_{C^1} - js)j = j - j = 0$ due to properties 2 and 3), $sr = 0$ (for similar reasons), and $-\partial^2 s \partial^1 r = s\partial^1 r \partial^0$.

Example 2 *Let (C^0, ∂^0), (C^1, ∂^1), (C^2, ∂^2) be the chain complexes depicted on Figure 3. Together with i, j, r and s defined below, we obtain an effective short exact sequence.*

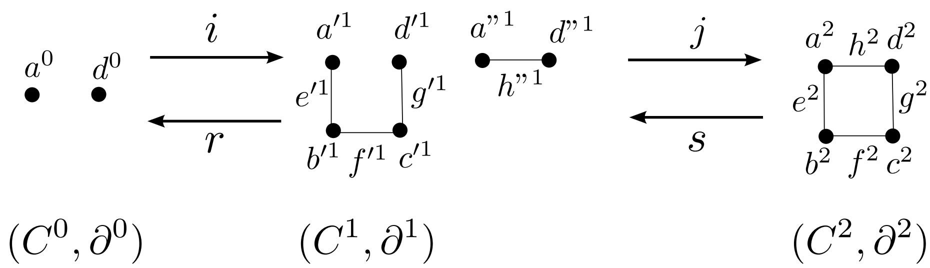


Figure 3: An Effective Short Exact Sequence

- $i : a^0 \to a'^1 - a"^1;\ d^0 \to d'^1 - d"^1;$
- $j : a'^1 \to a^2;\ a"^1 \to a^2;\ d'^1 \to d^2;\ d"^1 \to d^2;\ \forall x^1 \notin \{a'^1, a"^1, d'^1, d"^1\}$, $x^1 \to x^2;$
- $s : a^2 \to a'^1;\ d^2 \to d'^1;\ h^2 \to h"^1;\ \forall x^2 \notin \{a^2, d^2, h^2\}$, $x^2 \to x'^1;$
- $r : a'^1 \to 0;\ d'^1 \to 0;\ a"^1 \to -a^0;\ d"^1 \to -d^0;\ \forall x^1 \notin \{a'^1, a"^1, d'^1, d"^1\}$, $x^1 \to 0$.

□

Example 3 *Generalization of example 2.*
More generally, consider the following construction.

1. *Let $(C^1, \partial^1) = (C'^1, \partial'^1) \oplus (C"^1, \partial"^1)$, where C'^1 and $C"^1$ are respectively decomposed into two subsets P'^1, R'^1 and $P"^1, R"^1$, such that*[7]*:*
   - *$(P'^1, \partial'^1{}_{|P'^1})$ and $(P"^1, \partial"^1{}_{|P"^1})$ are subcomplexes,*
   - *$(P'^1, \partial'^1{}_{|P'^1})$ is isomorphic to $(P"^1, \partial"^1{}_{|P"^1})$.*

[7] In example 2, $P'^1 = \{a'^1, d'^1\}$, $R'^1 = \{e'^1, b'^1, f'^1, c'^1, g'^1\}$, $P"^1 = \{a"^1, d"^1\}$, $R"^1 = \{h"^1\}$.

2. *Let "glue" (C'^1, ∂'^1) and $(C"^1, \partial"^1)$ along P'^1 and $P"^1$, producing the chain complex $(C^2, \partial^2) = (C^1, \partial^1)/(P'^1 = P"^1)$[8]. This chain complex can be decomposed into P^2, R^2, such that the subcomplex $(P^2, \partial^2{}_{|P^2})$ is isomorphic to $(P'^1, \partial'^1{}_{|P'^1})$.*
3. *Let (C^0, ∂^0) be a chain complex isomorphic to $(P'^1, \partial'^1{}_{|P'^1})$. In order to simplify notations, if x^0 belongs to C^0 then x'^1, $x"^1$ are its images by the corresponding isomorphisms. Similar notations are also used for elements of P'^1, $P"^1$, and P^2, R'^1, $R"^1$ and R^2.*
4. *Let i, j, s, r be defined by:*
   - $i : \forall x^0 \in C^0, x^0 \to x'^1 - x"^1$;
   - $j : \forall x'^1 \in P'^1, x'^1 \to x^2$; $\forall x"^1 \in P"^1, x"^1 \to x^2$; $\forall x^1 \in R'^1 \cup R"^1, x^1 \to x^2$;
   - $s : \forall x^2 \in P^2, x^2 \to x'^1; \forall x^2 \in R^2, x^2 \to x^1$, *where* $x^1 \in R'^1 \cup R"^1$;
   - $r : \forall x'^1 \in P'^1, x'^1 \to 0$; $\forall x"^1 \in P"^1, x"^1 \to -x^0$; $\forall x^1 \in R'^1 \cup R"^1, x^1 \to 0$.

This construction is an effective short exact sequence, since:

- $\forall x^0 \in C^0$, $x^0 ir = x'^1 r - x"^1 r = 0 - (-x^0) = x^0$;
- $\forall x'^1 \in P'^1$, $x'^1(ri + js) = 0 + x^2 s = x'^1$;
  $\forall x"^1 \in P"^1$, $x"^1(ri + js) = -x^0 i + x^2 s = -(x'^1 - x"^1) + x'^1 = x"^1$;
  $\forall x^1 \in R'^1 \cup R"^1$, $x^1(ri + js) = 0 + x^2 s = x^1$;
- $\forall x^2 \in P^2$, $x^2 sj = x'^1 j = x^2$;
  $\forall x^2 \in R^2$, $x^2 sj = x^1 j = x^2$.

□

We give below only the subpart of the SES theorem which is useful for our constructions (see section 3 and section 4).

Theorem 2.14 *SES Theorem ([RS06] page 71)*
Let

$$0 \xrightarrow{0} (C^0, \partial^0) \underset{i}{\overset{r}{\leftrightarrows}} (C^1, \partial^1) \underset{j}{\overset{s}{\leftrightarrows}} (C^2, \partial^2) \xrightarrow{0} 0$$

be a short exact sequence together with two homological equivalences:

$$\Upsilon^0 : (C^0, \partial^0) \lll\!\ggg (C^{S0}, \partial^{S0})$$

$$\Upsilon^1 : (C^1, \partial^1) \lll\!\ggg (C^{S1}, \partial^{S1})$$

then it is possible to construct an homological equivalence Υ^2 relating (C^2, ∂^2) to a "small" chain complex constructed from (C^{S0}, ∂^{S0}) and (C^{S1}, ∂^{S1}).

[8] $(C^1, \partial^1)/(P'^1 = P"^1)$ denotes the quotient of (C^1, ∂^1) by the equivalence relation $P'^1 = P"^1$. (C^2, ∂^2) is thus the complex obtained by identifying the elements of P'^1 with the corresponding elements of $P"^1$.

Sketch of proof *(See Figure 4)*

More precisely let

$$\Upsilon^0 : (C^0, \partial^0) \overset{\rho^0}{\twoheadleftarrow\!\!\!\leftarrow} (C^{B0}, \partial^{B0}) \overset{\rho^{S0}}{\Rrightarrow} (C^{S0}, \partial^{S0})$$

and

$$\Upsilon^1 : (C^1, \partial^1) \overset{\rho^1}{\twoheadleftarrow\!\!\!\leftarrow} (C^{B1}, \partial^{B1}) \overset{\rho^{S1}}{\Rrightarrow} (C^{S1}, \partial^{S1})$$

be the two homological equivalences, with :

- $\rho^0 = ((C^{B0}, \partial^{B0}), (C^0, \partial^0), h^0, f^0, g^0)$,
- $\rho^{S0} = ((C^{B0}, \partial^{B0}), (C^{S0}, \partial^{S0}), h^{S0}, f^{S0}, g^{S0})$,
- $\rho^1 = ((C^{B1}, \partial^{B1}), (C^1, \partial^1), h^1, f^1, g^1)$
- $\rho^{S1} = ((C^{B1}, \partial^{B1}), (C^{S1}, \partial^{S1}), h^{S1}, f^{S1}, g^{S1})$.

Then two chain complex morphims, $i^B : (C^{B0}, \partial^{B0}) \longrightarrow (C^{B1}, \partial^{B1})$ and $i^S : (C^{S0}, \partial^{S0}) \longrightarrow (C^{S1}, \partial^{S1})$ can be deduced from the chain complex morphism i and both homological equivalences:

- $i^B = f^0 i g^1 : (C^{B0}, \partial^{B0}) \longrightarrow (C^{B1}, \partial^{B1})$;
- $i^S = g^{S0} i^B f^{S1} : (C^{S0}, \partial^{S0}) \longrightarrow (C^{S1}, \partial^{S1})$.

Moreover there exists an homological equivalence

$$\Upsilon^2 : (C^2, \partial^2) \overset{\rho^2}{\twoheadleftarrow\!\!\!\leftarrow} Cone(i^B) = (C^{B2}, \partial^{B2}) \overset{\rho^{S2}}{\Rrightarrow} Cone(i^S) = (C^{S2}, \partial^{S2})$$

Remember that $C^{B2} = C^{B0} \oplus C^{B1}$ and $C^{S2} = C^{S0} \oplus C^{S1}$. ρ^2 and ρ^{S2} are respectively defined by:

1. $\rho^2 = ((C^{B2}, \partial^{B2}), (C^2, \partial^2), h^2, f^2, g^2)$, *with:*
   - $h^2 = \begin{pmatrix} -h^0 & 0 \\ f^1 r g^0 & h^1 \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$
   - $f^2 = \begin{pmatrix} 0 \\ f^1 j \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^2$,
   - $g^2 = \begin{pmatrix} -s\partial^1 r g^0 & s g^1 \end{pmatrix} : C^2 \longrightarrow C^{B0} \oplus C^{B1}$,
2. $\rho^{S2} = ((C^{B2}, \partial^{B2}), (C^{S2}, \partial^{S2}), h^{S2}, f^{S2}, g^{S2})$, *with:*
   - $h^{S2} = \begin{pmatrix} -h^{S0} & h^{S0} i^B h^{S1} \\ 0 & h^{S1} \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$,
   - $f^{S2} = \begin{pmatrix} f^{S0} & h^{S0} i^B f^{S1} \\ 0 & f^{S1} \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{S0} \oplus C^{S1}$,
   - $g^{S2} = \begin{pmatrix} g^{S0} & -g^{S0} i^B h^{S1} \\ 0 & g^{S1} \end{pmatrix} : C^{S0} \oplus C^{S1} \longrightarrow C^{B0} \oplus C^{B1}$.

$$
\begin{tikzcd}
(C^0,\partial^0) \arrow[r, shift left, "i"] \arrow[d, shift left, "g^0"] & (C^1,\partial^1) \arrow[l, shift left, "r"] \arrow[r, shift left, "j"] \arrow[d, shift left, "g^1"] & (C^2,\partial^2) \arrow[l, shift left, "s"] \arrow[d, shift left, "g^2"] \\
(C^{B0},\partial^{B0}) \arrow[u, shift left, "f^0"] \arrow[d, shift left, "f^{S0}"] \arrow[r, "i^B"] \arrow[loop left, "h^0"] \arrow[loop right, "h^{S0}"] & (C^{B1},\partial^{B1}) \arrow[u, shift left, "f^1"] \arrow[d, shift left, "f^{S1}"] \arrow[loop left, "h^1"] \arrow[loop right, "h^{S1}"] & Cone(i^B) \arrow[u, shift left, "f^2"] \arrow[d, shift left, "f^{S2}"] \arrow[loop left, "h^2"] \arrow[loop right, "h^{S2}"] \\
(C^{S0},\partial^{S0}) \arrow[u, shift left, "g^{S0}"] \arrow[r, "i^S"] & (C^{S1},\partial^{S1}) \arrow[u, shift left, "g^{S1}"] & Cone(i^S) \arrow[u, shift left, "g^{S2}"]
\end{tikzcd}
$$

Figure 4: Effective short exact sequence theorem.

The complete constructions and proofs are recalled in Annex 6.5 page 58.
□

Example 4 *It is displayed on Figure 5 and comes from Example 2.*

Two homological equivalences are associated with (C^0, ∂^0) and (C^1, ∂^1): cf. the left and middle of Figure 5. Remember that $a^0 i = a'^1 - a"^1$, $d^0 i = d'^1 - d"^1$. i_B can be deduced from i and reductions ρ^0 and ρ^1, i.e. i_B: $a^{B0} \longrightarrow a'^{B1} - a"^{B1}$, $d^{B0} \longrightarrow d'^{B1} - d"^{B1}$. i_S can then be deduced from i_B and reductions ρ^{S0} and ρ^{S1}, i.e. i_S: $a^{S0} \longrightarrow a'^{S1} - a"^{S1}$, $d^{S0} \longrightarrow a'^{S1} - a"^{S1}$. Thus, by applying SES theorem, an homological equivalence involving (C^2, ∂^2), $Cone(i^B)$ and $Cone(i^S)$ can be deduced: cf. the right of Figure 5. Note that chain complexes are represented as geometric objects: for instance, on the right of Figure 5, a^{B0} is incident to a'^{B1} and $a"^{B1}$, meaning that the boundary of a^{B0} contains a'^{B1} and $a"^{B1}$ as generators (since a^{B0} corresponds to a^0, $a^0 i = a'^1 - a"^1$, and a'^1 and $a"^1$ corresponds to a'^{B1} and $a"^{B1}$).

□

3 Application to semi-simplicial sets

3.1 Semi-simplicial sets

Semi-simplicial sets are well-known in Algebraic Topology [May67]; they generalize abstract simplicial complexes, for which a similar approach has been

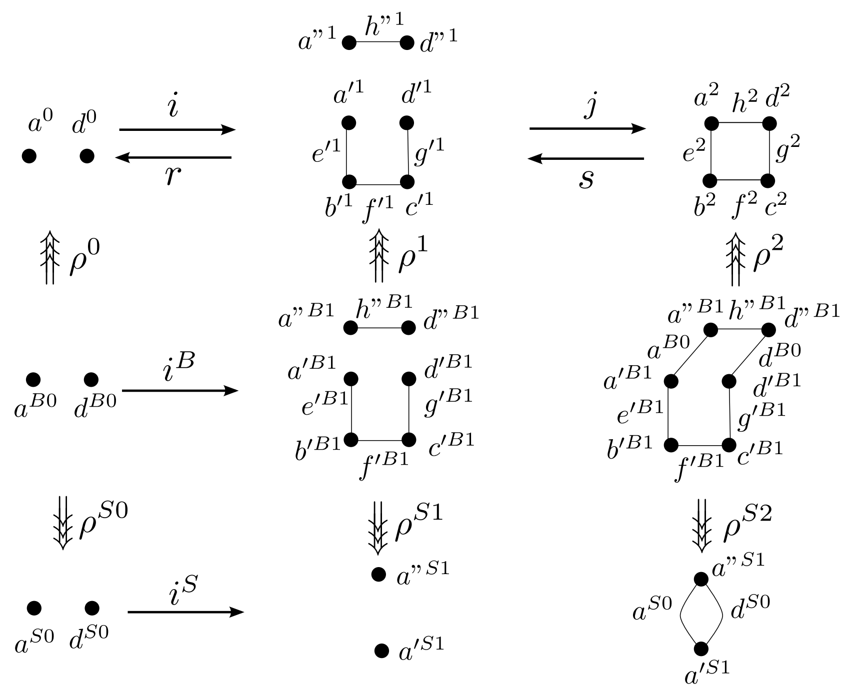


Figure 5: Illustration of SES theorem.

followed in [BMLH10, BCMA+11]. Moreover semi-simplicial sets are deeply related to generalized maps studied in section 4.

Definition 3.1 (semi-simplicial set) *An n-dimensional semi-simplicial set $S = (K, (d_j)_{j=0,\dots,n})$ is defined by (cf. figure 6):*

- *$K = \bigcup_{i=0,\dots,n} K_i$, where K_i is a finite set the elements of which are called i-simplices;*
- *$\forall j \in \{0,\dots,n\}$, $d_j : K \longrightarrow K$, called face operator, is s.t.:*
  - *$\forall i \in \{1,\dots,n\}, \forall j \in \{0,\dots,i\}$, $d_j : K_i \longrightarrow K_{i-1}$; $\forall j > i$, d_j is undefined on K_i, and no face operator is defined on K_0;*
  - *$\forall i \in \{2,\dots,n\}$, $\forall j,k \in \{0,\dots,i\}$, $d_j d_k = d_k d_{j-1}$ for $k < j$.*

Let σ be a simplex. A *proper face* τ of σ is the image of σ by a non empty composition of face operators; if $dim(\tau) = i$, τ is an *$i-$face* of σ. A simplex which is not the proper face of any simplex is a *main simplex*. The *boundary* of σ is the semi-simplicial set restricted to its proper faces. The *star* of σ is the set of simplices from which σ is a proper face. The *connected component* of σ is the semi-simplicial set obtained from σ by closure of the relations "boundary" and ”star”. A $i-$dimensional *complete* simplex is a $i-$simplex incident to $i+1$ vertices, together with its boundary.

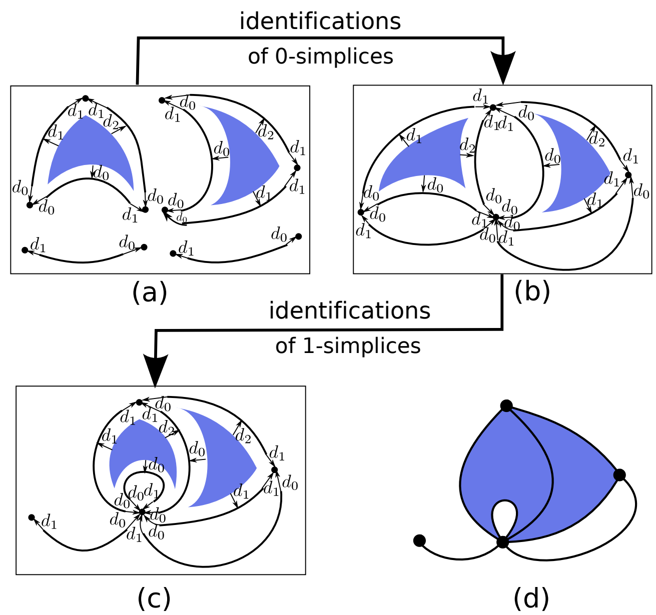


Figure 6: (c) A semi-simplicial set S (a) Complete simplices corresponding to the main simplices of S (b) and (c) S is constructed by applying identifications (d) the corresponding simplicial ”object”.

Cone and identification are basic operations, which are useful for constructing semi-simplicial sets [LL95, PFL09][9].

Definition 3.2 *Cone (cf. Figure 7)*
Let $S = (K, (d_j)_{j=0,\dots,n})$ *be a semi-simplicial set.* $S^c = (K^c, (d^c_j)_{j=0,\dots,n})$, *the cone of* S*, is defined by:*

- *Let* K' *be a copy of* K *by a one-to-one mapping* ϕ *which associates a* $(j+1)$*-simplex of* K' *with any* j*-simplex of* K*. Let* σ *be a* 0*-simplex which does belong neither to* K *nor to* K'*.*

$$K^c = K \cup K' \cup \{\sigma\}$$

- *Face operators are defined by:*

[9]Note that any semi-simplicial S set can be constructed by applying cone and identification operations: for instance, a possible way consists in first constructing complete simplices corresponding to the main simplices of S by using cone operation, and then gluing these simplices by identifying some proper faces in order to get S (see Figure 6).

  - *Let* $\mu \in K$, $\forall j$, $\mu d_j^c = \mu d_j$
  - *Let* $\mu \in K'_i$:
    * *if* $i = 1$, *then* $\mu d_1^c = \mu\phi^{-1}$ *and* $\mu d_0^c = \sigma$
    * *else* $\mu d_i^c = \mu\phi^{-1}$, *and* $\forall 0 \leq j < i$, $\mu d_j^c = \mu\phi^{-1} d_j \phi$

Note that any complete i-simplex can be constructed from a 0-simplex by applying i cone operations.

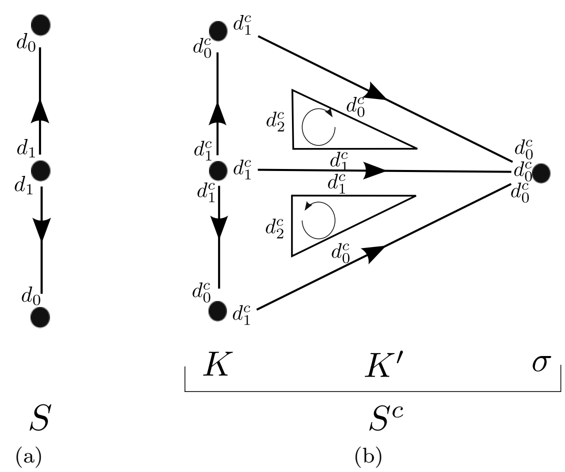


Figure 7: Cone operation applied to a complex K composed of three vertices and two edges.

Definition 3.3 *Identification (cf. Figure 6)*

Let $S = (K, (d_j)_{j=0,\dots,n})$ *be a semi-simplicial set, and let* σ *and* τ *be two* k*-simplices such that* $\sigma d_j = \tau d_j$, $\forall j$.

$S^i = (K^i, (d_j^i)_{j=0,\dots,n})$, *the identification of* σ *and* τ *in* S, *is defined by:*

- K^i *is defined by:*
  - $\forall j \neq k$, $K_j^i = K_j$;
  - $K_k^i = K_k - \{\sigma, \tau\} \cup \{\rho\}$, *where* ρ *is a* k*-simplex which does not belong to* K_k.
- *Face operators are defined by:*
  - $\forall h \neq k, k+1$, $\forall \mu \in K_h^i$, $\forall j$, $\mu d_j^i = \mu d_j$.
  - $\forall \mu \in K_{k+1}^i$, $\forall j$, *if* $\mu d_j = \sigma$ *or* $\mu d_j = \tau$ *then* $\mu d_j^i = \rho$ *else* $\mu d_j^i = \mu d_j$
  - $\forall \mu \in K_k^i$, $\forall j$, *if* $\mu \neq \rho$, $\mu d_j^i = \mu d_j$ *else* $\rho d_j^i = \sigma d_j$.

This operation can be easily extended in order to identify two simplices having different proper faces by identifying first these distinct proper faces and then the simplices themselves. It can then be extended in order to simultaneously identify a subset of simplices having the same dimension, and then again in order to identify several such subsets.

At last, note that a chain complex can be associated with any semi-simplicial set.

Definition 3.4 *The chain complex (C, ∂) associated with a semi-simplicial set $S = (K, (d_j)_{j=0,\ldots,n})$ is defined by:*

- $C = \bigcup_{i=0,\ldots,n} C_i$ *where* $\forall i$, C_i *is the group generated by* K_i*,*
- $\forall i > 0$, ∂_i *is linearly generated by:* $\forall \sigma \in K_i$, $\sigma\partial_i = \sum_{j=0}^{i}(-1)^j \sigma d_j$

3.2 Homological equivalences and basic operations

We study now the computation of a homological equivalence which can be associated with the result of a cone or identification operation. In the following, when complexity or implementation issues are discussed:

- we assume that the representation of the chain complex associated with a semi-simplicial set is such that:
  - each generator corresponds to a simplex;
  - the boundary of each generator corresponds to the boundary of its associated simplex;
- the complexity of a chain group C, denoted $|C|$, is the number of its generators;
- let Φ be a chain group morphism, the complexity of $\Phi(c)$ is the number of generators involved in the resulting chain. The complexity of Φ, denoted $|\Phi|$, is the sum of the complexities of Φ applied to all generators;
- the complexity of a chain complex (C, ∂), denoted $|C, \partial|$, is the sum of the complexities of C and ∂.

3.2.1 Cone operation

Theorem 3.5 *Let S^1 be a semi-simplicial set and (C^1, ∂^1) be its associated chain complex. Let S^2 be the cone of S^1 and (C^2, ∂^2) be its associated chain complex.*

Let $\Upsilon^2 : (C^2, \partial^2) \overset{\rho^2}{\Lleftarrow} (C^2, \partial^2) \overset{\rho^{S2}}{\Rrightarrow} (C^{S2}, \partial^{S2})$ *be such that (with the notations of definition 3.2):*

- $\rho^2 = ((C^2, \partial^2), (C^2, \partial^2), h^2 = 0, f^2 = id, g^2 = id)$

- $\rho^{S2} = ((C^2, \partial^2), (C^{S2}, \partial^{S2}), h^{S2}, f^{S2}, g^{S2})$, *with:*
  - C^{S2} *contains only one* 0*-dimensional generator* σ^2*;*
  - $\partial^{S2} = 0$*;*
  - *let* μ *be any generator of* C^2*; if* μ *corresponds to a simplex of* S^1*, then* $\mu h^{S2} = (-1)^{i+1}\mu\phi$*, where* i *is the dimension of* μ*, else* $\mu h^{S2} = 0$*;*
  - *let* ν *be any generator of* C^2*; if the dimension of* ν *is* 0*, then* $\nu f^{S2} = \sigma^2$*, else* $\nu f^{S2} = 0$*;*
  - $\sigma^2 g^{S2} = \sigma$.

Υ^2 *is an homological equivalence which can be computed in linear time according to* (C^1, ∂^1)*. The size of* Υ^2 *is linear according to* (C^1, ∂^1)*.*

The proof is in annex 6.6 page 62. Note that (C^{S2}, ∂^{S2}) is the homology of (C^2, ∂^2), since ∂^{S2} is null : it is a well-known result that the homology of a cone is trivial.

3.2.2 Identification operation

The following lemma is useful for the study of the identification operation (cf. example 5).

Lemma 3.6 *An effective short exact sequence can be associated with any identification, in linear time according to the size of the chain complex associated with the semi-simplicial set.*

Proof *We use here the notations of Definition 2.13.*

Let $S^1 = (K^1, (d_i^1)_{i=0,\dots,n})$ *be a semi-simplicial set and* (C^1, ∂^1) *its associated chain complex. Assume that identifications of simplices have to be performed on* S^1*. These identifications can involve other identifications in the boundary of these simplices. All these identifications can be easily deduced and we assume that all simplices to be identified are structured in the following way:* $I = \{I_1, I_2, \dots, I_k\}$ *where each* I_u *contains simplices to be identified together and if* σ_1^1 *and* σ_2^1 *belong to* I_u *then* $\forall p$*, either* $\sigma_1^1 d_p^1 = \sigma_2^1 d_p^1$ *or* $\exists q$*, such that* $\sigma_1^1 d_p^1 \in I_q$ *and* $\sigma_2^1 d_p^1 \in I_q$*.*

Let $S^2 = (K^2, (d_i^2)_{i=0,\dots,n})$ *be the result of all identifications and* (C^2, ∂^2) *be its associated chain complex. Note that* $|C^2, \partial^2| \leq |C^1, \partial^1|$*, and that its computation time is linear in* $|C^1, \partial^1|$ *(in practice, if* S^1 *is modified in order to produce* S^2*, the modification time is sub-linear).*

Below, (C^0, ∂^0)*,* i, j, r, s *are defined, so that:*

$$0 \xrightarrow{0} (C^0, \partial^0) \underset{i}{\overset{r}{\leftrightarrows}} (C^1, \partial^1) \underset{j}{\overset{s}{\leftrightarrows}} (C^2, \partial^2) \xrightarrow{0} 0$$

is a short exact sequence.

1. *Let j be the surjective mapping between S^1 and S^2, corresponding to the identification. More precisely, let σ^1 be a simplex of S^1, if σ^1 does not belong to I, then a simplex of S^2 is uniquely associated with σ^1; else u exists, such that $\sigma^1 \in I_u$, and all simplices of I_u are uniquely associated with their resulting simplex in S^2. Thus due to the definition of I, j is obviously a surjective chain-complex morphism. Moreover, $|j| = |C^1|$.*
2. *For each I_u, we choose a representative simplex denoted σ^1_u. Let s be the mapping defined by: let σ^2 be a simplex of S^2. Either σ^2 is the image by j of a unique simplex σ^1 and $\sigma^2 s = \sigma^1$, or there exists u such that σ^2 is the image by j of all simplices in I_u and $\sigma^2 s = \sigma^1_u$. s is obviously a graded module morphism, and $|s| = |C^2|$.*
3. *Let us define C^0, i, and ∂^0 by*
   - *for each $\sigma^1 \in I$ which is not a representative simplex, there exists in C^0 a generator σ^0 of same dimension[10]. So, $|C^0|$ is (almost) the number of identified simplices, and $|C^0| \leq |C^1|$;*
   - *for each $\sigma^0 \in C^0$, $\sigma^0 i = \sigma^1_u - \sigma^1$, where σ^0 corresponds to σ^1 and σ^1_u is the representative simplex of σ^1. i is obviously an injective graded module morphism, and $|i| = 2|C^0|$;*
   - *∂^0 is defined in order to satisfy: $\sigma^0\partial^0 i = \sigma^0 i\partial^1$ where σ^0 is a generator of C^0. As $\sigma^0 i = \sigma^1_u - \sigma^1$, $\sigma^0\partial^0 i = \sigma^1_u\partial^1 - \sigma^1\partial^1 = \sum_p (-1)^p(\sigma^1_u d^1_p - \sigma^1 d^1_p)$. For a given p:*
     - *either $\sigma^1_u d^1_p = \sigma^1 d^1_p$,*
     - *or q exists such that $\sigma^1_u d^1_p$ and $\sigma^1 d^1_p$ belong to I_q (see definition of I above). In this case:*
       - *either $\sigma^1_u d^1_p$ or $\sigma^1 d^1_p$ is the representative element of I_q and an element of C^0 corresponds to $\pm(\sigma^1_u d^1_p - \sigma^1 d^1_p)$,*
       - *or $\sigma^1_u d^1_p - \sigma^1 d^1_p$ is the image by i of the difference of two elements of C^0, as $\sigma^1_u d^1_p - \sigma^1 d^1_p = (\sigma^1_q - \sigma^1 d^1_p) - (\sigma^1_q - \sigma^1_u d^1_p)$ where σ^1_q is the representative element of I_q.*

     Since i is injective, it is thus possible to define $\sigma^0\partial^0 \in C^0$ in unique way, and $|\partial^0| \leq 2|\partial^1|$ (in fact, $|\partial^0|$ is at most twice the size of ∂^1 restricted to the identified simplices).
   - *∂^0 is a boundary operator, since $\forall \sigma^0 \in C^0$, $\sigma^0\partial^0\partial^0 i = \sigma^0 i\partial^1\partial^1 = 0$ and i is injective.*
   - *Thus (C^0, ∂^0) is a chain complex and i is a chain-complex morphism.*
4. *Let σ^1 be a simplex of S^1. Either σ^1 does not belong to I and $\sigma^1 r = 0$. Or σ^1 belongs to some I_u; let σ^1_u be the representative simplex of I_u: either $\sigma^1 = \sigma^1_u$ and $\sigma^1 r = 0$, or $\sigma^1 r = -\sigma^0$ where $\sigma^0 i = \sigma^1_u - \sigma^1$. r is obviously a graded module morphism, and $|r| = |C^0|$;*

[10] In other words, a graded module morphism exists between C^0 and the subset of I which contains the non representative simplices.

So, the computation of $(C^0, \partial^0), i, j, r$ *and* s *is linear in the size of* (C^1, ∂^1). *We check below the properties of* i, j, r, s.

- $\forall \sigma^0 \in C^0, \sigma^0 ir = (\sigma^1_u - \sigma^1)r = -\sigma^1 r = -(-\sigma^0) = \sigma^0$, *thus* $ir = id_{|C^0}$
- $\forall \sigma^1 \in S^1$,
  - *either* $\sigma^1 = \sigma^1_u$ *is a representative simplex of some* I_u *then* $\sigma^1_u ri + \sigma^1_u js = 0 + \sigma^2 s = \sigma^1_u$
  - *or* σ^1 *belongs to some* I_u, *such that* $\sigma^1_u \neq \sigma^1$ *is the representative simplex of* I_u. $\sigma^1 ri + \sigma^1 js = -(\sigma^1_u - \sigma^1) + \sigma^1_u = \sigma^1$
  - *or* σ^1 *does not belong to* I, *then* $\sigma^1 ri + \sigma^1 js = 0 + \sigma^1$

  Thus $ri + js = id_{|C^1}$.
- $\forall \sigma^2 \in S^2$, $\sigma^2 sj = \sigma^2$ *and thus* $sj = id_{|C^2}$.

□

Example 5 *Let* S^1 *be the semi-simplicial set depicted on Fig. 8(a). Assume edges* j^1 *and* n^1 *(resp.* p^1 *and* q^1, m^1 *and* r^1*) are identified into* j^2 *(resp.* p^2, m^2*). This involve the identification of vertices* a^1 *and* f^1 *(resp.* c^1, e^1 *and* h^1*) into* a^2 *(resp.* c^2*), producing the semi-simplicial set* S^2 *depicted on Fig. 8(b). Let:*

- $I = \{\{a^1, f^1\}, \{j^1, n^1\}, \{c^1, e^1, h^1\}, \{p^1, q^1\}, \{m^1, r^1\}\}$
- j *is defined by:* $a^1 j = f^1 j = a^2$; $j^1 j = n^1 j = j^2$; $c^1 j = e^1 j = h^1 j = c^2$; $p^1 j = q^1 j = p^2$; $m^1 j = r^1 j = m^2$; $b^1 j = b^2$; $d^1 j = d^2$; $g^1 j = g^2$; $i^1 j = i^2$; $k^1 j = k^2$; $l^1 j = l^2$; $o^1 j = o^2$; $s^1 j = s^2$; $t^1 j = t^2$; $u^1 j = u^2$; $v^1 j = v^2$; $w^1 j = w^2$;
- a^1, j^1, c^1, p^1, m^1 *are chosen as representative simplices. Thus* $a^2 s = a^1$; $j^2 s = j^1$; $c^2 s = c^1$; $p^2 s = p^1$; $m^2 s = m^1$; $b^2 s = b^1$; $d^2 s = d^1$; $g^2 s = g^1$; $i^2 s = i^1$; $k^2 s = k^1$; $l^2 s = l^1$; $o^2 s = o^1$; $s^2 s = s^1$; $t^2 s = t^1$; $u^2 s = u^1$; $v^2 s = v^1$; $w^2 s = w^1$;
- *Let* $C^0 = \{f^0, n^0, e^0, h^0, q^0, r^0\}$; *thus* $f^0 i = a^1 - f^1$; $n^0 i = j^1 - n^1$; $e^0 i = c^1 - e^1$; $h^0 i = c^1 - h^1$; $q^0 i = p^1 - q^1$; $r^0 i = m^1 - r^1$. $f^0 \partial^0 = 0$; $n^0 \partial^0 = f^0 - e^0$; $e^0 \partial^0 = 0$; $h^0 \partial^0 = 0$; $q^0 \partial^0 = h^0 - e^0$; $r^0 \partial^0 = h^0$. *More precisely:*
  - $n^0 i \partial^1 = (j^1 - n^1)\partial^1 = a^1 - c^1 - f^1 + e^1 = (a^1 - f^1) - (c^1 - e^1) = (f^0 - e^0)i$;
  - $q^0 i \partial^1 = (p^1 - q^1)\partial^1 = e^1 - g^1 - h^1 + g^1 = e^1 - h^1 = (c^1 - h^1) - (c^1 - e^1) = (h^0 - e^0)i$;
  - $r^0 i \partial^1 = (m^1 - r^1)\partial^1 = c^1 - d^1 - h^1 + d^1 = c^1 - h^1 = h^0 i$;
- $f^1 r = -f^0$; $n^1 r = -n^0$; $e^1 r = -e^0$; $h^1 r = -h^0$; $q^1 r = -q^0$; $r^1 r = -r^0$; *for any other simplex* σ, $\sigma r = 0$.

□

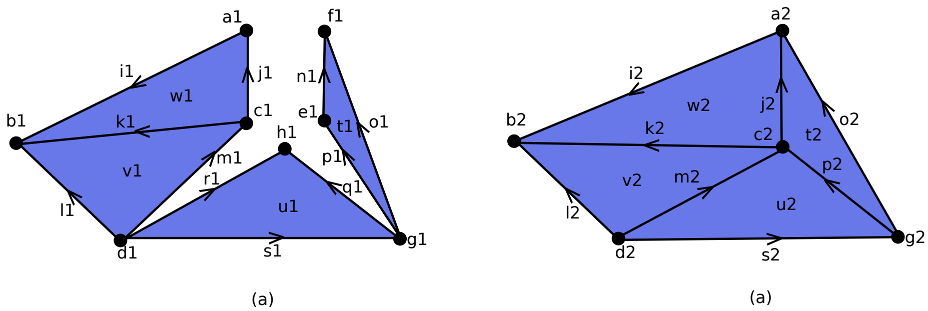


Figure 8: Identifications of several simplices of S^1 (left), producing S^2 (right)

Theorem 3.7 *Let S^1 be a semi-simplicial set, (C^1, ∂^1) be its associated chain complex and $\Upsilon^1 : (C^1, \partial^1) \overset{\rho^1}{\Lleftarrow} (C^{B1}, \partial^{B1}) \overset{\rho^{S1}}{\Rrightarrow} (C^{S1}, \partial^{S1})$ be a homological equivalence. Let S^2 be the result of any identification applied to S^1, and (C^2, ∂^2) be its associated chain complex. A homological equivalence $\Upsilon^2 : (C^2, \partial^2) \overset{\rho^2}{\Lleftarrow} (C^{B2}, \partial^{B2}) \overset{\rho^{S2}}{\Rrightarrow} (C^{S2}, \partial^{S2})$ can be deduced from Υ^1 and the identification.*

Proof *Let:*

- $(C^0, \partial^0), i, j, r, s$ *be defined as in the proof of lemma 3.6;*
- $\rho^0 = ((C^0, \partial^0), (C^0, \partial^0), h^0 = 0, f^0 = id, g^0 = id)$;
- $\rho^{S0} = ((C^0, \partial^0), (C^{S0}, \partial^{S0}), h^{S0}, f^{S0}, g^{S0})$ *be any reduction defined on* (C^0, ∂^0) *(at least a reduction exists :* $((C^0, \partial^0), (C^0, \partial^0), h^{S0} = 0, f^{S0} = id, g^{S0} = id)$*);*
- $\Upsilon^0 : (C^0, \partial^0) \overset{\rho^0}{\Lleftarrow} (C^0, \partial^0) \overset{\rho^{S0}}{\Rrightarrow} (C^{S0}, \partial^{S0})$;

By applying theorem 2.14, we get (using the notations of this theorem):

- $i^B : (C^0, \partial^0) \to (C^{B1}, \partial^{B1})$,
- $i^S : (C^{S0}, \partial^{S0}) \to (C^{S1}, \partial^{S1})$,
- *and* $\Upsilon^2 : (C^2, \partial^2) \overset{\rho^2}{\Lleftarrow} Cone(i^B) = (C^{B2}, \partial^{B2}) \overset{\rho^{S2}}{\Rrightarrow} Cone(i^S) = (C^{S2}, \partial^{S2})$.

□

Example 6 *Figure 9 shows two construction steps. The initial semi-simplicial set S contains two complete 2-dimensional simplices. Two edges incident to the upper triangle are identified during the first construction step. Then, the resulting edge is identified with an edge incident to the other 2-dimensional simplex.*

In fact, figure 9 shows the semi-simplicial sets together with (representations of) homological equivalences, and illustrates the application of theorem 3.7. The "big" complex of the homological equivalence associated with S is a chain complex which is isomorphic the chain complex associated with S; the "small" complex contains only two 0-dimensional generators, i.e. it is the homology of S. The first identification is described on the left of figure 9. The short exact sequence on the top is constructed as described in the proof of lemma 3.6; we have associated a homological equivalence with the chain complex top left. Then i^B, i^S and the homological equivalence associated with the semi-simplicial set resulting from the identification are computed as described in the proof of theorem 3.7. The second identification is described on the right of figure 9.

□

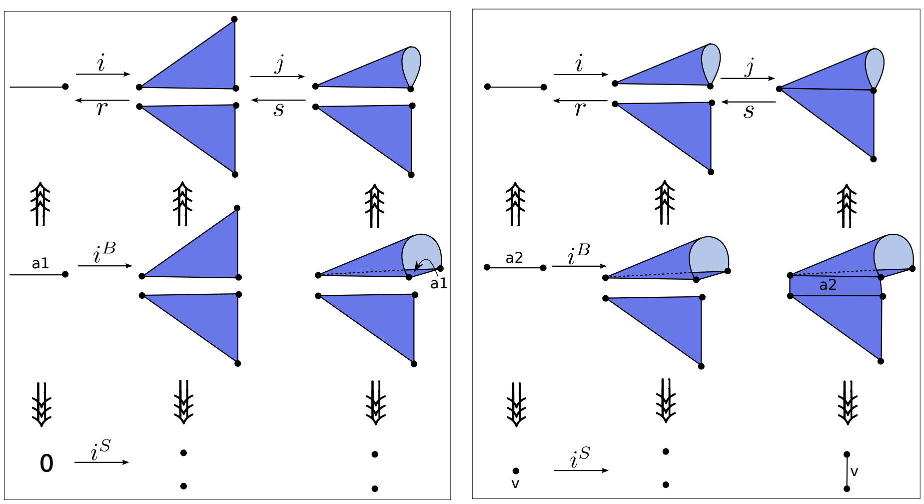


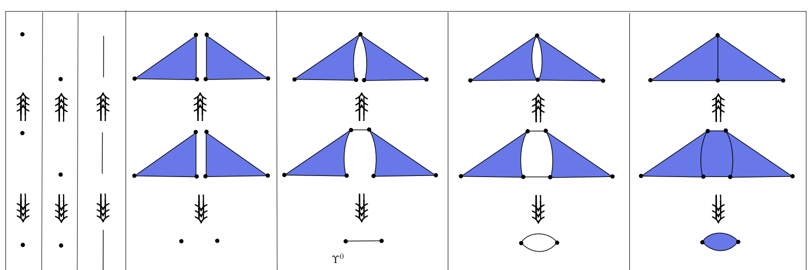
Figure 9: Incremental computation of homological equivalences

The complexity of the computation of the resulting homological equivalence Υ^2 is related to the complexity of Υ^1, and thus depends on the operations which have been applied to construct S^1: cf. corollary 6.9 in Annex 6.7. Cone and identification are basic operations, which make it possible to construct any semi-simplicial set: the study of these operations is thus interesting from a theoretical point of view. This is also interesting from a practical point of view, since these operations can be applied during a construction process; but this is not sufficient for concluding about the complexity related to the construction described in theorem 3.7, since other operations exist, which can be applied for constructing semi-simplicial sets.

Nevertheless, note that, when several identifications are performed on a semi-simplicial set, at most all i-dimensional simplices are identified into one i-simplex, for any i (and any further identification is impossible); thus the number of generators of the resulting "big" complex is at most twice the number of initial simplices, since any identification of two simplices "adds" a corresponding generator in the "big" complex. So, the number of generators in Υ^2 is at most twice the number of generators of Υ^1.

A second result is the following. Remember that any semi-simplicial set S can be constructed by cones and identifications: cf. figure 10. More precisely :

- associate a complete simplex with any main simplex of S, and let S' be the semi-simplicial set which contains all these complete simplices. A homological equivalence can thus be associated with any complete simplex, in which the "big" complex corresponds to the complete simplex and the "small" complex is made of one 0-dimensional generator. The homological equivalence associated with S' is the sum of all these homological equivalences;
- make the necessary identifications in order to get S, by increasing dimensions. For any identification, apply the construction described in theorem 3.7, where $\rho^{S0} = ((C^0, \partial^0), (C^0, \partial^0), h^{S0} = 0, f^{S0} = id, g^{S0} = id)$.



Identification 1	Υ^0			Υ^1	Υ^2		
Identification 2		Υ^0			Υ^1	Υ^2	
Identification 3			Υ^0			Υ^1	Υ^2

Figure 10: Construction of a semi-simplicial set by cones and identifications. An homological equivalence is computed at each step. The four columns top right describe homological equivalences corresponding to the initial semi-simplicial set and to the three semi-simplicial sets constructed by applying identifications. The colums top left describe, for the three identifications, the homological equivalences corresponding to the identified simplices. The three rows down denote the homological equivalences corresponding to the three identifications, with the notations of theorem 3.7.

It can be shown that the size and the computation time of the resulting homological equivalence Υ is linear in the size of S': cf. Annex 6.7. This bound is interesting from a theoretical point of view, but obviously, this construction is not very interesting for the computation of homology, since the number of generators of the "small" complex of Υ is the number of main simplices of S plus the number of identifications of two simplices applied to S' in order to get S (remember that when k simplices are identified into one simplex, $k-1$ generators are added).

In practice, when an identification is performed, simplifications, as applying elementary reductions (cf. annex 6.2) or computing a homological equivalence corresponding to a Smith Normal Form, can be applied at two moments (cf. Fig. 11):

- when computing homological equivalence Υ^0 (using the notations of theorem 3.7), and more precisely ρ^{s_0} (i.e. when (C^{S0}, ∂^{s_0}) is deduced from (C^0, ∂^0)); so it is interesting here to identify simultaneously simplices and simplices of their boundaries (such simplifications cannot be applied when identifying simplices by increasing dimensions, for instance);
- after Υ^2 has been computed by applying theorem 3.7, it could be possible, and interesting, to simplify its "small" complex, since Υ^2 at one construction step is Υ^1 at the next construction step.

Of course, such simplifications induce more computing time, and it can thus be interesting to elaborate a strategy, according to the constructions which are performed.

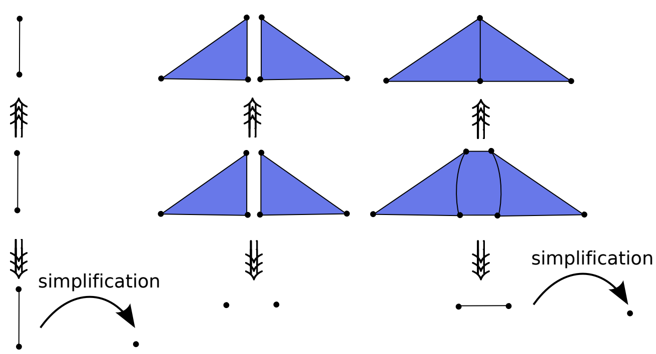


Figure 11: Two kinds of simplifications allowing to reduce the small complex of the resulting homological equivalence. In this example, $2*3$ simplices are identified in the same step. The small complexes of Υ^0 and Υ^2 can be simplified.

Note that for low dimensions, look-up tables can be precomputed in order to store homological equivalences corresponding to identification configurations:

- for main simplices and their boundaries for obtaining the initial Υ^1 (cf. Fig. 12(a))
- for identified simplices for obtaining Υ^0 at each step (cf. Fig. 12(b)).

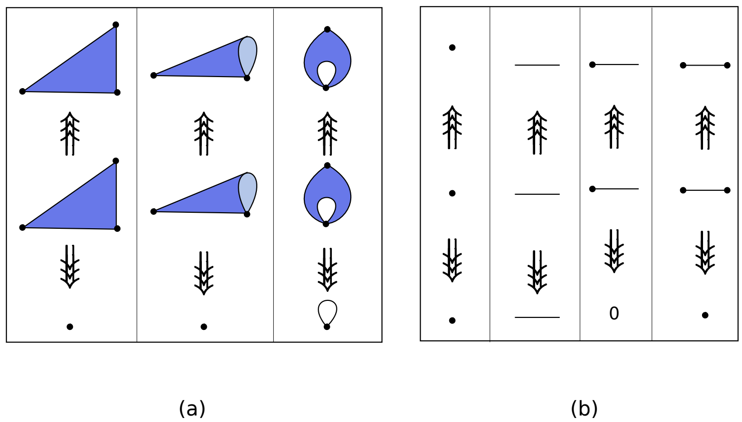


Figure 12: (a) a part of the lookup table regarding main triangles and their boundaries. (b) lookup table for identified simplices of dim 0 and 1.

Note that these results can be applied to:

- abstract simplicial complexes [Ago76]; in this case, the identification operation has to be restricted in order to avoid multi-incidence, i.e. in order to avoid the degeneracy of simplices;
- simplicial sets [May67]; in this case, no modification is necessary, since homology does not depend upon degeneracy operators.

At last, note that tools and results of Effective Homology have also been applied by Boltcheva *et al* [BMLH10] for the conception of an algorithm called *Meyer-Vietoris (MV) algorithm*, which computes the homological information of abstract simplicial complexes from the homological information of their sub-complexes and their intersections. This algorithm has been applied for the *Manifold-Connected (MC) decomposition* of abstract simplicial complexes [BCMA+11]. More precisely, a MC decomposition of an abstract simplicial complex A is a decomposition of A into sub-complexes which satisfy combinatorial properties close to that of quasi-manifolds (cf. Section 4.1.2). Given two sub-complexes B and C, the MV algorithm computes the homological information of $B \cup C$ from the homological informations of B and C, and the homological information of $B \cap C$.

Although the same tools and results are applied here, the approach described in this section is different, even if it is restricted to abstract simplicial complexes. For instance, assume two simplices incident to the same connected component are identified, and that the initial homological information of the connected component is known: the construction described in the proof of theorem 3.7 can be applied. But without other information, it is not possible to efficiently apply the MV algorithm in order to compute the homological information of the resulting abstract simplicial complex, since no decomposition is taken into account. Conversely, let A and B be two abstract simplicial complexes, such that $A \cap B$ is an abstract simplicial complex. Let A' (resp. B') be a copy of A (resp. B) such that A'_{AB} (resp. B'_{AB}) is the sub-complex of A' (resp. B') corresponding to $A \cap B$. Then it is possible to compute the homological information of $A \cup B$ either by the MV algorithm, or by identifying A'_{AB} and B'_{AB} and applying theorem 3.7. In this way, theorem 3.7 generalizes the approach of Boltcheva *et al*, since the identification operation is a very basic construction operation.

4 Application to generalized maps

In section 4.1, notions related to generalized maps and their equivalence with a subclass of semi-simplicial sets are recalled. An incremental method for computing their homology, based upon the results of sections 2 and 3, is proposed in section 4.2.

4.1 Generalized maps and their simplicial interpretation

This section is mainly based on [Lie94] and [DL14].

4.1.1 Generalized maps and their canonical construction

Definition 4.1 (n-gmap) *Let $n \geq 0$, an n-gmap is defined by an $(n+2)$-tuple $G = (D, \alpha_0, \cdots, \alpha_n)$ such that:*

- *D is a finite set of darts;*
- *$\forall i, 0 \leq i \leq n, \alpha_i : D \to D$ is an involution;*
- *$\forall i, 0 \leq i \leq n-2, \forall j, i+2 \leq j \leq n, \alpha_i\alpha_j$ is an involution.*

By convention, $G = (D)$ is a $(-1)-$gmap, where D is a set of darts.

The notion of *orbit* is fundamental for gmaps since many notions, as cells, incidence and adjacency are based on.

Definition 4.2 (orbit) *Let $\Phi = \{\pi_0, \cdots, \pi_n\}$ a set of permutations defined on a set D. The permutation group of D generated by Φ is denoted $\langle\Phi\rangle = \langle\pi_0, \cdots, \pi_n\rangle = \langle\rangle_{[0,n]}$. The orbit of an element $d \in D$ relatively to $\langle\Phi\rangle$, denoted $\langle\Phi\rangle(d)$, is the set $\{d\phi \mid \phi \in \langle\Phi\rangle\}$. It denotes also the structure $(D^d = \langle\Phi\rangle(d), \pi_0/D^d, \cdots, \pi_n/D^d)$, where π_i/D^d denotes the restriction[11] of π_i to D^d.*

[11] We often omit to explicitly indicate the restriction, since it is usually obvious.

Let $G = (D, \alpha_0, \cdots, \alpha_n)$ be a $n-$gmap. A connected component of G is an orbit relatively to $\langle \alpha_0, \cdots, \alpha_n \rangle$ (also denoted $<>_{[0,n]}$ *or* $<>_N$). An $i-$dimensional cell of G is an orbit relatively to $\langle \alpha_0, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n \rangle$ (also denoted $<>_{N-\{i\}}$).

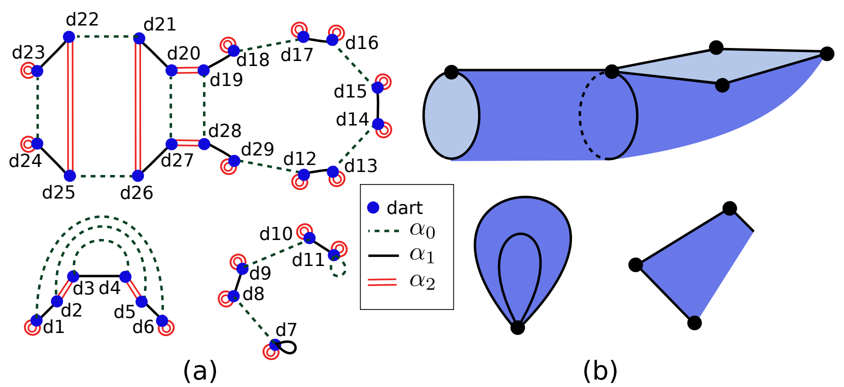


Figure 13: Graphical representation of a 2-gmap: the conventions for representing darts and involutions are described between (a) and (b). (a) a 2-gmap made of 3 connected components. (b) The corresponding "cellular objects".

For example the gmap represented in figure 13 contains 3 connected components. On the top, the connected component contains 2 faces: a square face $< \alpha_0, \alpha_1 > (d26) = \{d_{20}, \cdots, d_{27}\}$, and a pentagon $< \alpha_0, \alpha_1 > (d28) = \{d_{12}, \cdots, d_{19}, d_{28}, d_{29}\}$. These 2 faces share the edge $< \alpha_0, \alpha_2 > (d19) = \{d_{19}, d_{20}, d_{27}, d_{28}\}$. Each face is incident only once to this edge since each face and the edge share a single orbit relatively to α_0. Similarly the square face makes a cylinder since it is incident twice to the edge $< \alpha_0, \alpha_2 > (d21) = \{d_{21}, d_{22}, d_{25}, d_{26}\}$ (the face and the edge share 2 orbits relatively to α_0 i.e. $< \alpha_0 > (d_{21})$ and $< \alpha_0 > (d_{26})$).

Any $n-$gmap $G = (D, \alpha_0, ..., \alpha_n)$ can be obtained from the $(-1)-$gmap $G = (D)$ by applying *extension*, and *sewing* operations[12], defined below (cf. figure 14).

Definition 4.3 *The extension of* $G = (D, \alpha_0, ..., \alpha_n)$ *is the* $(n+1)-$*gmap* $G' = (D, \alpha_0, ..., \alpha_n, \alpha_{n+1} = id)$.

Definition 4.4 *Let* $G = (D, \alpha_0, \cdots, \alpha_n)$, $d_1, d_2 \in D$ *such that:*

- $d_1 \neq d_2$,

[12] Gmaps can be handled through other sets of operations [Lie94, BSP+05, DL03, DD09, BPA+10].

- $d_1\alpha_n = d_1$ *and* $d_2\alpha_n = d_2$,
- *an isomorphism* ϕ *exists between* $O_{d1} =< \alpha_0, ..., \alpha_{n-2} > (d_1)$ *and* $O_{d2} =< \alpha_0, ..., \alpha_{n-2} > (d_2)$,
- *if* $O_{d1} = O_{d2}$, *then* $\phi = \phi^{-1}$.

The sewing of d_1 *and* d_2 *by* α_n *is the gmap* $G' = (D, \alpha_0, \cdots, \alpha_{n-1}, \alpha'_n)$, *where*[13]*:*

- $\forall d \in D - (O_{d1} \cup O_{d2})$, $d\alpha'_n = d\alpha_n$
- $\forall d \in O_{d1}, d\alpha'_n = d\phi$
- *if* $O_{d1} \neq O_{d2}$, *then* $\forall d \in O_{d2}, d\alpha'_n = d\phi^{-1}$.

Gmaps *without self-bending* are constructed by exclusively sewing distinct orbits (this corresponds to $O_{d1} \neq O_{d2}$ in definition 4.4).

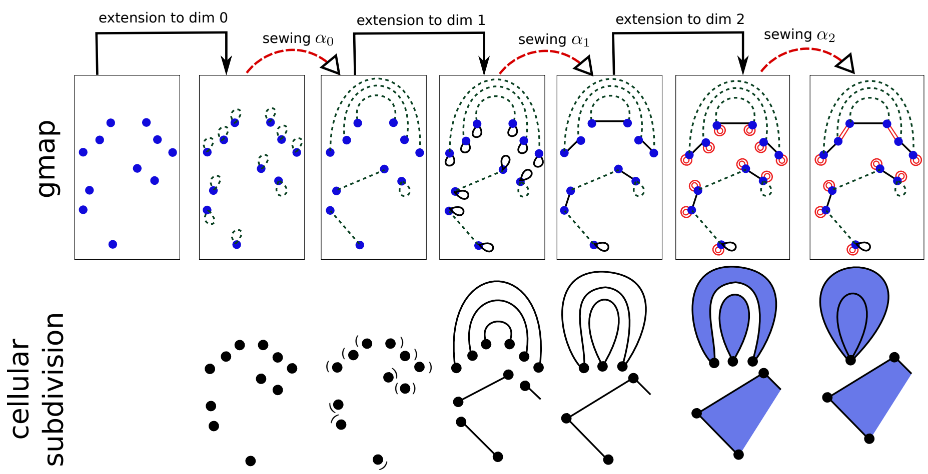


Figure 14: The 2−gmap $G' = (D, \alpha_0, \alpha_1, \alpha_2)$ on the right is obtained from the (-1)−gmap $G = (D)$ on the left by applying a sequence of extension and sewing operations. The corresponding "cellular object" is represented under each gmap. Note that identifying two edges consists in sewing two isomorphic orbits relatively to α_0.

4.1.2 Cellular quasi-manifolds

Any gmap is equivalent to a semi-simplicial set structured into cells by a numbering of 0-simplices. Here the notion of *numbered semi-simplicial sets* is recalled, then two subclasses of numbered semi-simplicial sets are distinguished, in particular *cellular quasi-manifolds*, which are equivalent to gmaps.

[13]Note that a general sewing operation is defined for any $n-$gmap, and any dimension $i, 0 \leq i \leq n$ [Lie94].

Definition 4.5 *A numbered n-dimensional semi-simplicial set* $S = (K, (d_j)_{j=0,\dots,n}, \nu)$ *is a n-dimensional semi-simplicial set together with a mapping* $\nu : K^0 \to \{0, \cdots, n\}$ *such that each main simplex* $\sigma \in K$ *satisfies (cf. Fig. 15):*

- *if* $dim(\sigma) = 0$ *then* $\sigma\nu = 0$
- *if* $dim(\sigma) = i > 0$ *then its numbering, i.e. the sequence of integers associated with its* $0-$*faces by* ν*, is* $(0, \cdots, i)$

Remark 1 *It is always possible to define the face operators s.t., for each* $k-$*simplex* σ *numbered* $(i_0, \cdots, i_j, \cdots, i_k)$*,* $i_0 < \cdots < i_j < \cdots < i_k$*,* σd_j *is numbered* $(i_0, \cdots, i_{j-1}, i_{j+1}, \cdots, i_k)$*.*

Definition 4.6 *An* $i-$*cell is a vertex numbered* i *together with the subset of its star made of the simplices which numberings have* i *as highest integer.*

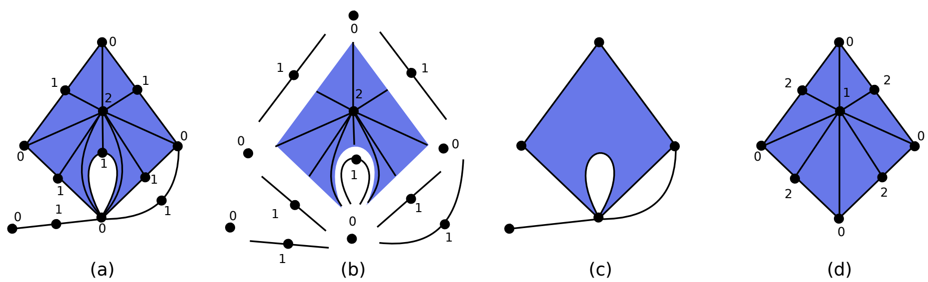


Figure 15: (a) A numbered semi-simplicial set. A simplified representation is used, since the face operators can be retrieved, thanks to remark 1. (b) The cells. (c) The corresponding "cellular object". (d) A numbered simplicial quasi-manifold which is not a cellular quasi-manifold, since the $1-$cell is incident to 4 vertices.

Definition 4.7 *A numbered* $n-$*dimensional simplicial quasi-manifold is a numbered* $n-$*dimensional semi-simplicial set, which can be constructed by (cf. Fig. 15(d)):*

- *adding* $n-$*simplices numbered* $(0, \cdots, n)$ *and their boundaries.*
- *identifying* $(n-1)-$*simplices (and their boundaries), in such a way that at most 2* $n-$*simplices belong to the star of any* $(n-1)-$*simplex.*

Definition 4.8 (Cellular quasi-manifold) *A* n*-dimensional cellular quasi-manifold is (cf. figure 16):*

1. *for* $n = 0$*, a* $0-$*dimensional numbered simplicial quasi-manifold made of a set of* $0-$*simplices partitioned into connected components made of* 1 *or* 2 $0-$*simplices*[14]*;*

[14] A connected component containing 2 $0-$simplices describes the topology of a $0-$sphere.

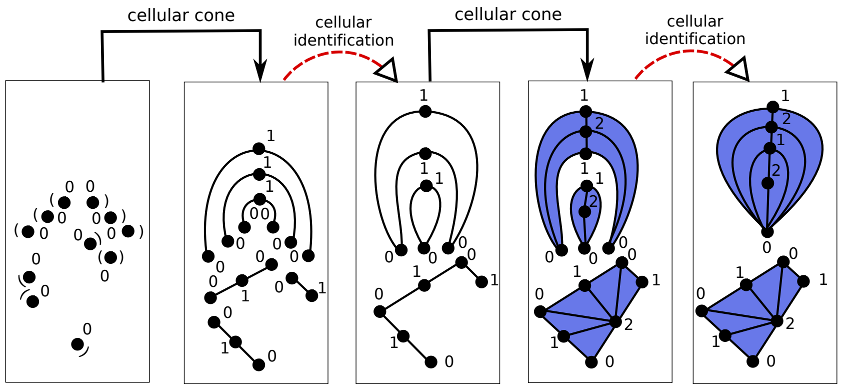


Figure 16: Construction of the cellular quasi-manifold corresponding to the gmap represented in figure 14, applying cellular cones and cellular identifications.

2. *for $n \geq 1$, either:*
   - a $n-$dimensional cellular cone, *i.e. the cone of a connected $(n-1)-$dimensional cellular quasi-manifold*[15] *where the new vertex is numbered n;*
   - *or* the result of the cellular identification *of* 2 *$(n-1)$-cells c_1 and c_2 of a n-dimensional cellular quasi-manifold, s.t.:*
     - *each $(n-1)-$simplex of c_1 and c_2 is incident to one $n-$simplex,*
     - *an isomorphism ϕ exists between c_1 and c_2. If $c_1 = c_2$, then ϕ is an automorphism different from the identity s.t. $\phi = \phi^{-1}$.*

   The cellular identification consists in identifying the $(n-1)-$simplices (and their boundaries) of c_1 and c_2 according to ϕ.

Note that contrary to numbered simplicial quasi-manifolds, any cell of a cellular quasi-manifold is the inside of a cellular quasi-manifold (cf. Fig. 15(d) for a counter-example).

Theorem 4.9 *Generalized maps are equivalent to cellular quasi-manifolds.*

The conversion processes between a $n-$gmap G and its corresponding cellular quasi-manifold M are (cf. Figs. 14 and 16):

- $M \rightarrow G$: a dart is associated with each $n-$simplex ; two darts corresponding to two $n-$simplices sharing a $(n-1)-$simplex numbered $(0, \cdots, i-1, i+1, \cdots, n)$ are linked by α_i; in fact, all α_i correspond to adjacency relations between $n-$simplices of M. When $n = 0$, α_0 links darts corresponding to $0-$simplices belonging to the same connected component.

[15]Note that the cellular cone contains only one $n-$cell.

- $G \to M$: Let $N = [0, n]$. For any non-empty $I \subseteq N$, a $(\mid I \mid -1)-$simplex[16] is associated with any orbit $<>_{N-I}$; for any j, for any σ corresponding to an orbit $<>_{N-I}$ (b), σd_j is the simplex corresponding to $<>_{N-I'}$ (b), where I' is I without its j^{th} element. When $n = 0$, each connected component of M corresponds to a connected component of G.

Note that the extension and sewing operations for gmaps are equivalent to cellular cone and cellular identification operations for cellular quasi-manifolds[17].

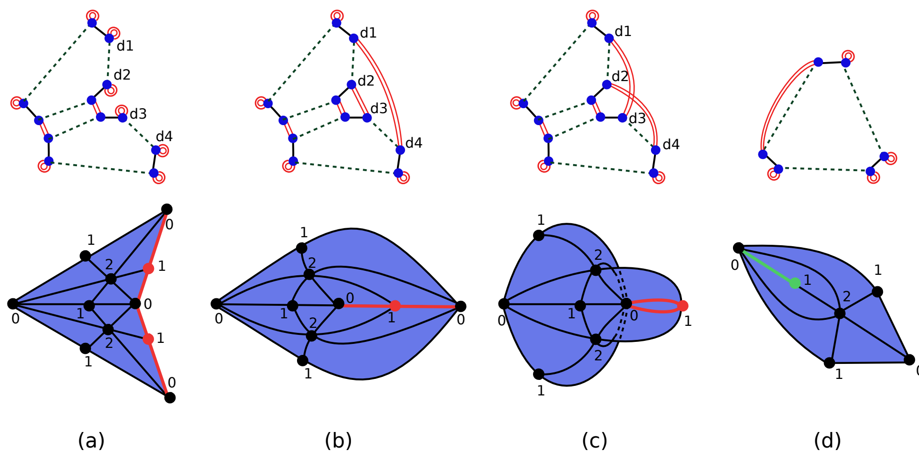


Figure 17: (a) a gmap and its corresponding cellular quasi manifold. (b) On the top, sewing of two isomorphic orbits of dimension 1 (i.e. two edges), s.t. $d1\phi = d4$ and $d2\phi = d3$. On the bottom, the corresponding identifications in the associated cellular quasi-manifold. (c) On the top, sewing of two isomorphic edges s.t. $d1\phi = d3$ and $d2\phi = d4$. On the bottom, the corresponding identifications in the associated cellular quasi-manifold. Note that in this case, 3 $0-$cells are identified. (d) a gmap with self-bending: an edge is identified with itself. In the associated cellular quasi-manifold, the corresponding $1-$cell is not isomorphic to other edges.

4.2 Homological equivalences and basic operations

In the following, only gmaps without self bending are considered (cf. 4.2.4 for the generalization to any gmap).

A chain complex can be associated with any gmap. It is the chain complex associated with the cellular quasi-manifold corresponding to the gmap. So, and as for semi-simplicial sets, each generator of the chain complex corresponds to

[16]$\mid I \mid$ denotes the number of elements of I.

[17]More generally, all important notions can be directly translated: for instance, the notions of cells in cellular quasi-manifolds can be directly retrieved through the notion of compacted cells of gmaps: cf. property 5, [ADLP15].

a simplex of the cellular quasi-manifold, and the boundary of each generator corresponds to the boundary of the associated simplex. Moreover, a homological equivalence can be associated with any gmap, and the results presented in Section 3.2 can be directly applied to gmaps, since extension corresponds to cone operations and sewing corresponds to simplex identifications.

In order to take advantage of the cell structuration of a cellular quasi-manifold, it can be interesting to associate homological equivalences not only with gmaps *but also with their cells*. So, when cells are sewn together, the homological equivalences associated with the identified cells can be used in order to efficiently compute the homological equivalences of the resulting gmap.

Note that such homological equivalences can be associated with any 0-gmap. Let $G = (D, \alpha_0)$ be a 0-gmap. Its corresponding cellular quasi-manifold is obtained by associating a vertex numbered 0 with each dart. An homological equivalence can be built for each connected component and for each $0-$cell. For each homological equivalence, the 3 chain complexes are isomorphic, homotopy operator h is nul, f and g mappings correspond to the isomorphisms. Figure 18 illustrates the homological equivalences for a 0-gmap made of 3 darts in 2 connected components.

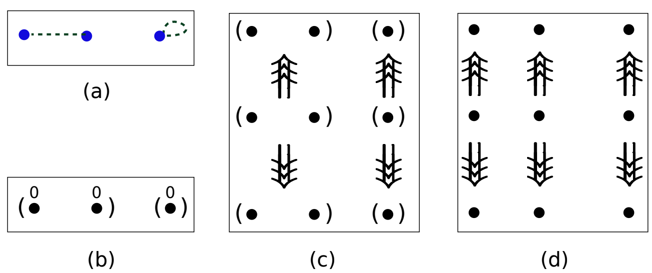


Figure 18: (a) a $0-$gmap, (b) its corresponding cellular quasi-manifold, (c) homological equivalences related to the connected components, (d) homological equivalences related to the cells.

Now, we show that it is possible to incrementally compute homological equivalences (for the resulting gmap and its cells) when extension or sewing is applied. This is interesting from a theoretical point of view, since any gmap can be constructed, starting with a 0-gmap, by applying these operations: this shows that such homological equivalences can be incrementally computed for any gmap. Moreover, this is also interesting from a practical point of view, since extension and sewing are useful basic operations for construction processes.

In the following, a gmap (resp. its corresponding cellular quasi-manifold, its corresponding homological equivalences) will be denoted by G (resp. M, U).

4.2.1 Extension

Theorem 4.10 *Given a $(n-1)$-gmap $G^1 = (D, \alpha_0, \cdots, \alpha_{n-1})$, its cellular quasi-manifold M^1 and its homological equivalences U^1, it is possible to compute the cellular quasi-manifold M^2 and the homological equivalences U^2 of G^2, the extension of G^1, in linear time according to the size of M^1 and U^1 respectively.*

Proof *The cellular quasi-manifold M^2, associated to G^2 is obtained by applying cellular cones on each connected component of M^1: the size of M^2 is thus linear according to the size of M^1, and its computation time is also linear in the size of M^1 (cf. definition 3.2 and Fig. 16). In a first step, we explain how to obtain homological equivalences for the connected components of M^2. In a second step, we explain how to obtain homological equivalences for the cells of M^2.*

Homological equivalences for connected components

For each connected component cc^1 of M^1, the connected component cc^2 corresponding to its extension is obtained by applying a cone over cc^1: theorem 3.5 can thus be applied. More precisely, the associated homological equivalence Υ^2 is the following: (C^{B2}, ∂^{B2}) is isomorphic to (C^2, ∂^2), and (C^{S2}, ∂^{S2}) is made of a single vertex with a null boundary operator since it is a well known result that the homology of a cone is trivial. The computation of (C^2, ∂^2) is linear in the size of cc^1 since it is a cone. So, the same result holds for (C^{B2}, ∂^{B2}), and the computation time of (C^{S2}, ∂^{S2}) is constant. The two reductions $\rho^2 : C^{B2} \to C^2$ and $\rho^{S2} : C^{B2} \to C^{S2}$ are defined as follows:

- ρ^2 *: the homotopy operator h^2 is null, the mappings f^2 and g^2 are isomorphisms. Thus, the time complexity for computing ρ^2 is linear in the size of cc^1.*
- ρ^{S2} *: the homotopy operator h^{S2} associates with each $k-$simplex $\sigma \in cc^1$, the corresponding $(k+1)-$simplex τ in the cone, s.t. $\sigma h^{S2} = (-1)^{k+1}\tau$, h^{S2} is null everywhere else. The mapping f^{S2} is null everywhere except for the vertex of the cone and the vertices of cc^1, which are associated with the single vertex of C^{S2}. The mapping g^{S2} associates the vertex of C^{S2} with the vertex of the cone in C^{B2}. Thus, the time complexity for computing ρ^{S2} is linear in the size of cc^1.*

Homological equivalences for cells

For each dimension lower than n, cells are not modified, nor their homological equivalences. The computation time is thus linear in the size of U^1 restricted to these cells[18].

[18]In practice, nothing is changed: so, the modification time is null.

Let cc^1 be a connected component of M^1, and c be the $n-$cell of cc^2 obtained from the extension of cc^1. Let

$$\Upsilon^1 : (C^1, \partial^1) \overset{\rho^1}{\Lleftarrow} (C^{B1}, \partial^{B1}) \overset{\rho^{S1}}{\Rrightarrow} (C^{S1}, \partial^{S1})$$

be the homological equivalence associated with cc^1. The homological equivalence

$$\Upsilon^2 : (C^2, \partial^2) \overset{\rho^2}{\Lleftarrow} (C^{B2}, \partial^{B2}) \overset{\rho^{S2}}{\Rrightarrow} (C^{S2}, \partial^{S2})$$

associated with c is deduced from Υ^1 in the following way:

- *The new complex (C^2, ∂^2) is defined by $C^2 = C^2_0 \oplus \cdots \oplus C^2_n$, where:*
  - *C^2_0 has only one generator corresponding to the vertex v of the cone;*
  - *for $i \geq 1$, C^2_i is isomorphic to C^1_{i-1}. Let ϕ be the sum of all these isomorphisms.*

  The boundary operator ∂^2 is defined as $\partial'^2 + \delta^2$ where:
  - *for any $i, \partial'^2_i = \phi^{-1}\partial^1_{i-1}\phi$,*
  - *δ^2 is null everywhere, except δ^2_1 which associates v with any generator of C^2_1.*

  Note that ∂'^2 is a boundary operator. Moreover ∂^2 is a boundary operator since:
  - *$\forall i \neq 1, \partial^2_i = \partial'^2_i$;*
  - *remember that any 1-simplex of a cellular quasi-manifold is incident to two distinct 0-simplices ; so, and also due to the definition of a cone, for any generator $\sigma \in C^2_2$, $\sigma\partial^2 = e - e'$ where $e \neq e'$ and $e\partial^2_1 = e'\partial^2_1 = v$. So, δ^2 is a perturbation of ∂'^2.*

  The computation time of (C^2, ∂^2) is linear in the size of cc^1. In practice, the main changes are related to the boundary of 1-simplices of the n-cell c, so the modification time is linear in the number of 0-simplices of cc^1.

- *The new complex (C^{B2}, ∂^{B2}) is defined by $C^{B2} = C^{B2}_0 \oplus \cdots \oplus C^{B2}_n$, where:*
  - *C^{B2}_0 has only one generator v^B,*
  - *for any $i \geq 1$, C^{B2}_i is isomorphic to C^{B1}_{i-1}. Let ϕ^B be the sum of all these isomorphisms.*

  Let ∂'^{B2} be defined by: for any $i, \partial'^{B2}_i = (\phi^B)^{-1}\partial^{B1}_{i-1}\phi^B$. Since ∂^{B1} is a boundary operator, ∂'^{B2} is also a boundary operator.
  Let $\rho = ((C^{B2}, \partial'^{B2}), (C^2, \partial'^2), h, f, g)$ be such that:
  - *$h = (\phi^B)^{-1}h^1\phi^B$,*
  - *$f/(C^{B2} - \{v^B\}) = (\phi^B)^{-1}f^1\phi$, $v^B f = v$,*

– $g/(C^2 - \{v\}) = \phi^{-1} g^1 \phi^B$, $vg = v^B$,

where h^1, f^1, g^1 are the homotopy operator and morphisms of ρ^1. Since ρ^1 is a reduction, ρ is also a reduction.
Since δ^2 is a perturbation of ∂'^2, the easy perturbation lemma applies (cf. lemma 2.11). So:

– $\delta^{B2} = f\delta^2 g$ is a perturbation of ∂'^{B2}, and thus $\partial^{B2} = \partial'^{B2} + \delta^{B2}$ is a boundary operator,
– $\rho^{B2} = ((C^{B2}, \partial^{B2}), (C^2, \partial^2), h, f, g)$ is a reduction.

The computation time of C^{B2}, ∂^{B2} and the application of the easy perturbation lemma are linear in the size of cc^1. In practice, the main changes are related to the boundary of generators of C_1^{B2}. Thus the modification time is linear in the number of generators of C_0^{B1}.

- *The new complex (C^{S2}, ∂^{S2}) is defined by $C^{S2} = C_0^{S2} \oplus \cdots \oplus C_n^{S2}$, where:*
  – C_0^{S2} has only one generator v^S,
  – for any $i \geq 1$, C_i^{S2} is isomorphic to C_{i-1}^{S1}. Let ϕ^S be the sum of all these isomorphisms.

  Let ∂'^{S2} be defined by: for any $i, \partial_i'^{S2} = (\phi^S)^{-1} \partial_{i-1}^{S1} \phi^S$. Since ∂^{S1} is a boundary operator, ∂'^{S2} is also a boundary operator.
  Let $\rho^S = ((C^{B2}, \partial'^{B2}), (C^{S2}, \partial'^{S2}), h^S, f^S, g^S)$ be such that:
  – $h^S = (\phi^B)^{-1} h^{S1} \phi^B$,
  – $f^S/(C^{B2} - \{v^B\}) = (\phi^B)^{-1} f^{S1} \phi^S$, $v^B f = v^S$,
  – $g^S/(C^{S2} - \{v^S\}) = (\phi^S)^{-1} g^1 \phi^B$, $v^S g = v^B$,

  where h^{S1}, f^{S1}, g^{S1} are the homotopy operator and morphisms of ρ^{S1}. Since ρ^{S1} is a reduction, ρ^S is a reduction.
  Moreover, δ^{B2} is a perturbation of ∂'^{B2} s.t. $\delta^{B2} h^S = 0$, and thus δ^{B2} satisfies the nilpotency hypothesis: indeed, δ^2 is null everywhere except on C_1^2; so δ^{B2} is null everywhere except on C_1^{B2}, since $\delta^{B2} = f\delta^2 g$. Let γ be a chain of C_1^{B2}. $\gamma\delta_1^{B2} h^S = \alpha v g h^S = \alpha v^B h^S = 0$, since δ^2 associates v with any generator of C_1^2.
  So, ∂^{S2} and the reduction ρ^{S2} can be deduced from ρ^S by applying the basic perturbation lemma (cf. lemma 2.12), where the mappings Φ and Ψ are equal to the identity.
  The computation time of C^{S2}, ∂'^{S2} and the application of the basic perturbation lemma are linear in the number of the generators of C^{B1} (this is due to the fact that $\delta^{B2} h^S = 0$). In practice, the main changes are related to the boundary of generators of C_1^{S2}. Thus the modification time is linear in the number of generators of C_0^{S1}.

Complexity

Each homological equivalence restricted to a connected component or to a cell can be computed in linear time and space, according to the initial *corresponding homological equivalence, so the whole complexity of U^2 is linear in the size of U^1.*

In practice, it is only necessary to compute homological equivalences for:

- *the connected components: this is linear in the size of M^1,*
- *the n-cells: this is linear in the size of the corresponding homological equivalences of U^1.*

□

4.2.2 Sewing

Theorem 4.11 *Let $G^1 = (D, \alpha_0, \cdots, \alpha_n)$ be an $n-$gmap, and $d_1, d_2 \in D$ such that:*

- $d_1\alpha_n = d_1$ *and* $d_2\alpha_n = d_2$,
- $O_{d1} =< \alpha_0, ..., \alpha_{n-2} > (d_1) \neq O_{d2} =< \alpha_0, ..., \alpha_{n-2} > (d_2)$,
- *an isomorphism ϕ exists between O_{d1} and O_{d2}, matching d_1 onto d_2*

Let M^1 be its associated cellular quasi-manifold and U^1 be its homological equivalences. Let G^2 be the result of the sewing of d_1 and d_2, and M^2 be its associated cellular quasi-manifold. Homological equivalences U^2 associated with M^2 can be directly deduced from U^1, by one application of theorem 2.14.

Proof *Remember that the gmap is without self-bending. Since O_{d1} and O_{d2} are distinct, the sewing of d_1 and d_2 produces a gmap without self-bending. Note that this sewing corresponds in M^1 to the identification of the two distinct $(n-1)-$cells c_{d1} and c_{d2} associated with O_{d1} and O_{d2}, together with the identifications of the corresponding cells of their boundaries, for instance:*

- *In figure 16, the cellular identification of the two edges induces the identification of the three extremity vertices into one vertex.*
- *In figure 17(a), before their identification, the two edges share a common vertex, and their two other extremity vertices are distinct. In figure17(b), the common vertex is not changed by the identification, but the two other extremity vertices are identified into one vertex.*

So, it is possible to compute M^2 in linear time according to the size of M^1. Moreover, since O_{d1} and O_{d2} are isomorphic, c_{d1} and c_{d2} are isomorphic; if cells of their boundaries have to be identified into one cell, then all these cells (including the resulting cell) are isomorphic, since G^1 is without self-bending (cf. property 5 in [ADLP15]).

Homological equivalences for cells

Let c be a cell of M^2:

- *either c corresponds to a cell c' of M^1, i.e. c' is not affected by the identification; so the homological equivalence associated with c is a copy of the one of c';*
- *either c results from the identification of several cells of M^1; since all these cells are isomorphic, a copy of the homological equivalence associated with any of the initial cells can be associated with c. In practice, it should be interesting to choose the most "efficient" homological equivalence, for instance according to the size of its small chain complex.*

So the computation time is linear in the size of U^1 restricted to the cells. In practice nothing is modified, so the modification time is null.

Homological equivalences for connected components

Preliminary remarks and notations

Let

$$\Upsilon^1 : (C^1, \partial^1) \xLeftarrow{\rho^1} (C^{B1}, \partial^{B1}) \xRightarrow{\rho^{S1}} (C^{S1}, \partial^{S1})$$

be the homological equivalence associated with (the connected components of) $M^1 = (K^1, (d_i^1)_{i=0,\ldots,n})$, and thus with G^1. Let (C^2, ∂^2) be the chain complex associated with M^2, and thus with G^2.

Remember that the sewing of d_1 and d_2 of G^1 corresponds to the identification of cells c_{d1} and c_{d2} of M^1, together with the identification of cells of their boundaries if necessary. All these cells can be easily deduced from G^1 (by a simple traversal starting from O_{d1} and O_{d2}) or from M^1 (also by a simple traversal starting from c_{d1} and c_{d2}). In the following, "cell" denotes either the set of simplices, the set of simplices together with the internal boundary relations, the corresponding generators of a chain complex or the corresponding sub chain complex.

The cells which have to be identified can be structured in the following way: $I = \{I_1, I_2, \ldots, I_k\}$, where each I_u contains all cells which have to be identified together. Moreover, for each I_u, we choose a representative cell c_u; let ϕ_u be the surjective morphism which maps any non representative cell of I_u (i.e. different from c_u) onto c_u.

So, if c_1 and c_2 belong to I_u, c_1 and c_2 are isomorphic ; let $\phi_{1,2}$ be the corresponding isomorphism. Let $\sigma_1 \in c_1$ and $\sigma_2 \in c_2$ be two p-simplices, such that $\sigma_1\phi_{1,2} = \sigma_2$ and $p > 0$. So, either $\sigma_1 d_p = \sigma_2 d_p$, either $\sigma_1 d_p$ and $\sigma_2 d_p$ belong to two isomorphic cells of the same I_q, such that $\sigma_1 d_p$ and $\sigma_2 d_p$ are associated by the corresponding isomorphism. At last, remember that for any $j, 0 \leq j \leq p-1$, $\sigma_1 d_j$ (resp. $\sigma_2 d_j$) belongs to c_1 (resp. c_2), *and thus $\sigma_1 d_j \phi_{1,2} = \sigma_2 d_j$.*

Construction of a homological equivalence Υ^0

$$\Upsilon^0 : (C^0, \partial^0) \xLeftarrow{\rho^0} (C^{B0}, \partial^{B0}) \xRightarrow{\rho^{S0}} (C^{S0}, \partial^{S0})$$

Let

$$\Upsilon'^0 : (C^0, \partial'^0) \xLeftarrow{\rho'^0} (C^{B0}, \partial'^{B0}) \xRightarrow{\rho'^{S0}} (C^{S0}, \partial'^{S0})$$

be the sum of all homological equivalences associated with[19] the non representative cells of I. Let $\phi_{0,1}$ be the bijection between C^0 and I minus all its representative cells. For each cell of C^0, $\phi_{0,1}$ restricted to this cell is an isomorphism with the corresponding cell of C^1.

Let δ^0 be defined in the following way. For any p-dimensional generator σ of C^0, such that $p > 0$, let $\sigma^1 = \sigma\phi_{0,1}$ and $\sigma^1_u = \sigma^1\phi_u$:

- *if $\sigma^1_u d_p = \sigma^1 d_p$, then $\sigma\delta^0 = 0$;*
- *else v exists, such that $\sigma^1_u d_p$ and $\sigma^1 d_p$ belong to I_v. In this case:*
  - *if $\sigma^1_u d_p$ belongs to the representative cell of I_v, $\sigma^1 d_p$ does not belong to this representative cell, and a generator σ_s exists in C^0, such that $\sigma_s\phi_{0,1} = \sigma^1 d_p$; in this case, $\sigma\delta^0 = (-1)^p\sigma_s$;*
  - *conversely, if $\sigma^1 d_p$ belongs to the representative cell of I_v, $\sigma^1_u d_p$ does not belong to this representative cell, and a generator σ_r exists in C^0, such that $\sigma_r\phi_{0,1} = \sigma^1_u d_p$; in this case, $\sigma\delta^0 = -(-1)^p\sigma_r$;*
  - *at last, if neither $\sigma^1_u d_p$ nor $\sigma^1 d_p$ belong to the representative cell of I_v, let σ_r (resp. σ_s) be the generator of C^0 associated with $\sigma^1_u d_p$ (resp. $\sigma^1 d_p$); in this case, $\sigma\delta^0 = (-1)^p(\sigma_s - \sigma_r)$;*

Note that, if σ belongs to a homological equivalence corresponding to an identified p-dimensional cell, $\sigma\delta^0$ contains generators belonging to homological equivalences corresponding to cells of dimension lower than p. Moreover, δ^0 is a perturbation of ∂'^0. Let $i : C^0 \to C^1$ be defined by: for any generator σ of C^0, $\sigma i = \sigma^1_u - \sigma^1$. i is thus an injective graded module morphism. Let σ be a p-dimensional generator of C^0. $\sigma(\partial'^0 + \delta^0)i = \sigma\partial'^0 i + \sigma\delta^0 i$. Since $\phi_{0,1}$ and ϕ_u, restricted to cells, are isomorphisms, $\sigma\partial'^0 i = \sum_{j=0}^{p-1}(\sigma i)d_j = \sum_{j=0}^{p-1}(\sigma^1_u - \sigma^1)d_j$. It is easy to deduce from the four cases of the definition of δ^0 that $\sigma(\partial'^0 + \delta^0)i = \sigma i\partial^1$. Since i is injective, $\partial^0 = \partial'^0 + \delta^0$ is a boundary operator, and δ^0 is a perturbation of ∂'^0.

[19] or equivalent homological equivalences, since the homological equivalences associated with isomorphic cells are equivalent; in other words, for each u between 1 and k, we choose one "best" homological equivalence for each I_u, and Υ'^0 contains q copies of this homological equivalence, q being the number of non representative cells of I_u.

Reduction $\rho^0 : (C^0, \partial^0) \overset{\rho^0}{\Lleftarrow} (C^{B0}, \partial^{B0})$ can be deduced from reduction $\rho'^0 : ((C^{B0}, \partial'^{B0}), (C^0, \partial'^0), h^0, f^0, g^0)$ and δ^0 by applying the easy perturbation lemma, where $\partial^{B0} = \partial'^{B0} + \delta^{B0}$ and $\delta^{B0} = f^0\delta^0 g^0$.

Reduction $\rho^{S0} : (C^{B0}, \partial^{B0}) \overset{\rho^{S0}}{\Rrightarrow} (C^{S0}, \partial^{S0})$ can be deduced from reduction $\rho'^{S0} : ((C^{B0}, \partial'^{B0}), (C^{S0}, \partial'^{S0}), h^{S0}, f^{S0}, g^{S0})$ and δ^{B0} by applying the basic perturbation lemma. Indeed, it is easy to show that δ^{B0} satisfies the nilpotency hypothesis. Let c be a generator of C^{B0}: so, c belongs to some homological equivalence corresponding to an identified p-dimensional cell, and cf^0 belongs to the the same homological equivalence. If cf^0 is not null, let c' be a generator of cf^0. Then $c'\delta^0$ is null, or it is a sum of generators of C^0: in this case, each of these generators belongs to a homological equivalence corresponding to a cell of lower dimension than p (cf. definition of δ^0 above), as the generators of $c'\delta^0 g^0$ and of $c'\delta^0 g^0 h^{S0}$. So, m exists, such that $c(\delta^{B0}h^{S0})^m = 0$; moreover, $m \leq n$, the dimension of the gmap.

Construction of an effective short exact sequence

1. *A surjective chain complex morphism $j : (C^1, \partial^1) \to (C^2, \partial^2)$ can be deduced from the sewing of d_1 and d_2; in particular, given u between 1 and k, all cells of I_u are mapped onto a unique resulting cell.*
2. *Let s be the mapping defined in the following way. Let c^2 be a cell of C^2, so:*
   - *either c^2 is the image by j of a unique cell c^1, and $c^2 s = c^1$;*
   - *or u exists, such that c^2 is the image by j of all cells of I_u, and $c^2 s$ is the representative cell of I_u.*

   Mapping s is a graded module morphism (in fact, restricted to each cell, it is a sub chain complex morphism).
3. *$i : C^0 \to C^1$ has been defined in the previous paragraph, when proving that δ^0 is a perturbation of ∂'^0; it has also been proved that i is an injective chain complex morphism;*
4. *$r : C^1 \to C^0$ associates any cell with 0, except the non representative cells of I: let c be such a cell, then $cr = -c\phi_{0,1}^{-1}$.*

As for the simplicial case, it is easy to prove that $(C^0, \partial^0), (C^1, \partial^1), (C^2, \partial^2)$, j, s, i, r define an effective short exact sequence.

Conclusion and complexity

It is clear that the SES theorem can be applied in order to deduce a homological equivalence

$$\Upsilon^2 : (C^2, \partial^2) \overset{\rho^2}{\Lleftarrow} (C^{B2}, \partial^{B2}) \overset{\rho^{S2}}{\Rrightarrow} (C^{S2}, \partial^{S2})$$

associated with M^2 and G^2, from Υ^0, Υ^1, j, s, i, r.

Note that all constructions are linear in the size of their arguments, except the application of the basic perturbation lemma (for the construction of Υ^0), which depends also upon the dimension of the identified cells, i.e. upon n, the dimension of the gmap.

$\square$

4.2.3 Remarks

1. As for the simplicial case, simplification techniques as elementary reductions or Smith Normal Form can be applied at any moment in order to reduce the size of the small complex of any homological equivalence (associated with a connected component or a cell). More precisely:

   - For the extension operation. It is not possible to simplify the homological equivalences associated with the connected components; the homological equivalences associated with cells of dimensions lower than n are not modified; so, it is not useful to try to simplify them, if it has been done at a construction step before. But it can be useful to try to simplify the homological equivalences associated with the n-cells: remember that (the homological equivalence Υ_c associated with) any n-cell c is deduced from (the homological equivalence Υ_{cc} associated with) a preexisting $(n-1)$-connected component cc. Even if it is not possible to simplify Υ_{cc}, it could be possible to simplify Υ_c: cf. Fig. 19;
   - For the sewing operation. The homological equivalences associated with cells after sewing are equal to homological equivalences associated with cells before sewing: so, it is not useful to try to simplify them, if it has been done at a construction step before. For the connected components, and as usual, it can be interesting to try to simplify the homological equivalence Υ^2 resulting from the application of the SES theorem. Moreover, it can be interesting also to try to simplify Υ^0 before the application of the SES theorem, since it results from the "linking" of all homological equivalences associated with the identified cells according to the boundary relations of these cells. This is similar to the simplicial case, when simplices and simplices of their boundaries are simultaneously identified: simplifications are possible, which are not possible when simplices of different dimensions are identified independently (cf. Fig. 9 and Fig. 10).

2. The construction of Υ^2 described in the proof of theorem 4.11 is based on the construction of Υ^0 and one application of the SES theorem. One application of the basic perturbation lemma is necessary in order to construct Υ^0, involving a complexity which is linear in the size of the homological

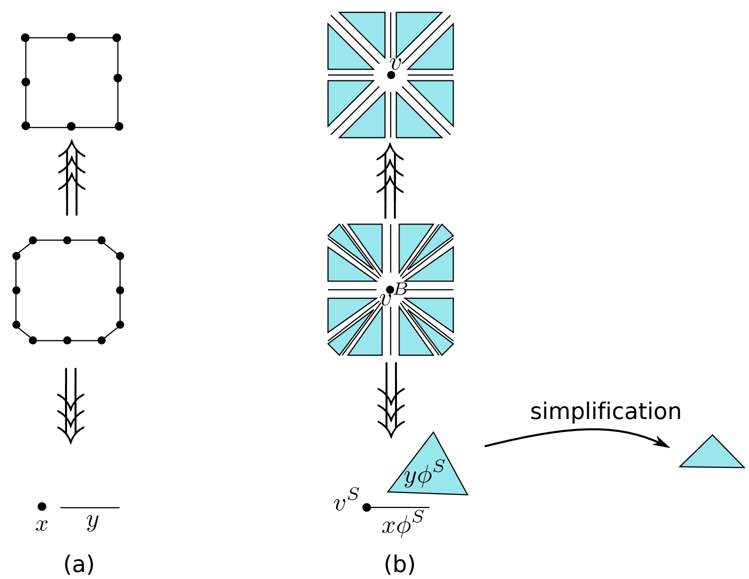


Figure 19: (a) an homological equivalence corresponding to a connected component. (b) The homological equivalence of the 2−cell deduced from (a) by the extension operation.

equivalences associated with the identified cells times their dimensions[20]. Note that it could be possible to construct a homological equivalence associated with M^2 in n steps, by identifying cells from dimension 0 until dimension $n-1$: SES theorem would be thus applied n times, but the complexity would not be better, and the possibility of simplifying Υ^0 before applying the SES theorem would be lost.

3. At last, if the number of "types" of cells is low, lookup tables can be used in order to memorize simplified homological equivalences which can then be used later for the identification of cells of the same types. In practice, when a homological equivalence has been computed, it is memorized in order to be used for the other isomorphic cells.

4. For low dimensions, lookup tables can also be used in order to memorize homological equivalences corresponding to identification configurations (i.e. corresponding to Υ^0). For instance in dimension 2, there exist one configuration for the identification of 2 vertices, and a few number of configurations for the identification of 2 edges[21]. For dimension 3, the

[20]This corresponds to the application of the Basic Perturbation Lemma: cf. nilpotency hypothesis.

[21]For instance, assume the two edges contain two darts each: so two possible isomorphisms exist, i.e. two ways for sewing the two edges. The different sewing configurations are distinguished by taking also into account the different configurations for the incident vertices: is an

number of configurations for the identification of 2-cells can be low, depending on a particular application.

When lookup tables are used, efficient methods should be used for computing isomorphisms of parts of gmaps (cf. [SDdJ15] for instance).

4.2.4 Generalizations

Extension for gmap with self-bending

When two darts belonging to the same $(n-1)$-cell are sewn together, the resulting $(n-1)$-cell is bent onto itself (cf. figure 20). Isomorphism ϕ of definition 4.4 is then an automorphism such that $\phi = \phi^{-1}$. Moreover, the bending of a cell on itself can induce the bending on themselves of cells of its boundary. So, it is necessary to compute:

- a homological equivalence for each cell c bent on itself; it can be computed by applying the SES theorem to c (since almost the "half" of c is identified with the other "half"), but it is necessary to compute a homological equivalence Υ_c^0 corresponding to the identified part. We don't know now how to take into account the homological equivalence associated with c in order to compute Υ_c^0, but Υ_c^0 can be computed as described in Subsection 3.2 (cf. proof of lemma 3.6 and theorem 3.7);
- a homological equivalence for the resulting connected components. The principle is the same as that described in the proof of theorem 4.11; but for each cell c bent on itself, it is necessary to use the homological equivalence Υ_c^0 computed previously instead of the homological equivalence associated with c.

Extension for oriented maps

Oriented maps is an optimization of gmaps for representing orientable cellular quasi-manifolds [Lie94, DL14]. Schematically, if a cellular quasi-manifold is orientable, the corresponding gmap "contains" the two possible orientations, and each orientation can be represented by an oriented map. Since each orientation is the inverse of the other, there is a strong correspondence between the two corresponding oriented maps. So, only one of these oriented maps is necessary in order to retrieve the corresponding gmap, and thus to represent the corresponding cellular quasi-manifold. So, all results presented above for gmaps can be directly applied to oriented maps, due to this correspondence between gmaps and oriented maps.

Extension for chains of maps

Chains of maps, or cmaps, is an extension of gmaps for representing cellular complexes such that each cell is a cell of a cellular quasi-manifold [EL95].

edge incident to one or two vertices ? are vertices incident to both edges ?

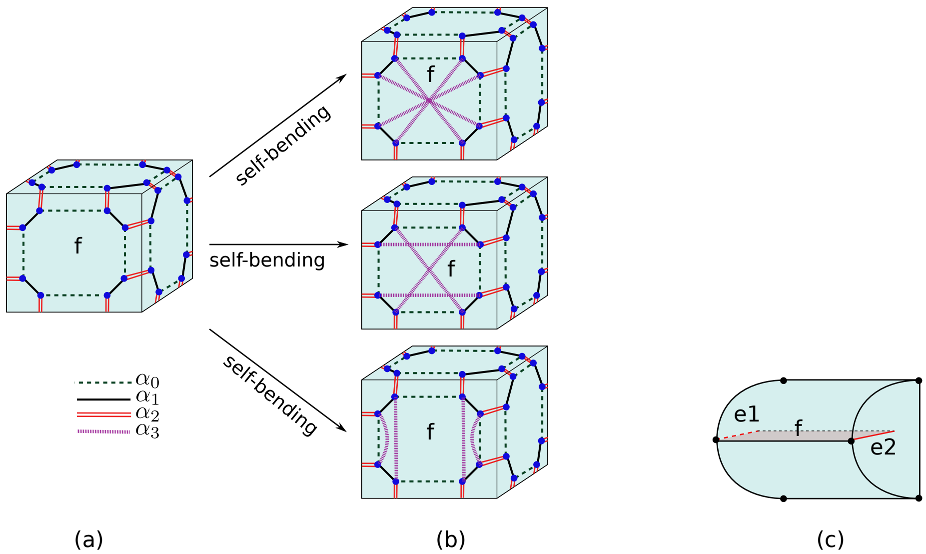


Figure 20: (a) a 3−gmap encoding a cube (α_3 is not represented, since $\alpha_3 = identity$). (b) three possible identifications of face f with itself (α_3 is represented only for the darts of f). In the bottom case, the self-bending of f induces the self-bending of edges of its boundary. (c) a geometric representation of the self-bending of f (bottom case): the two edges $e1$ and $e2$ of the boundary of f are also bent on themselves.

Schematically, a n-dimensional cmap is a set of i-gmaps, for $0 \leq i \leq n$, where each connected component of a i-gmap is a i-cell of the cmap; i-cells and $(i-1)$-cells of their boundaries are linked by face operators (cf. figure 21). The results obtained for gmaps can be easily extended for gmaps satisfying the condition described in [ADLP15], definition 4. Such a cmap can be constructed by:

- adding cells and their boundaries;
- identifying isomorphic i-cells, for any i: note that, as for gmaps, the identification of isomorphic i-cells can induce the identification of isomorphic cells of their boundaries.

As for gmaps, a homological equivalence can be associated with any cell, since a cell corresponds to a gmap; so all results presented for gmaps hold. A homological equivalence can also be associated with any connected component: initially, a connected component is made by one cell and its boundary, so all results presented above for gmaps apply here. The identification of i-cells can induce the identification of j-cells of their boundaries, with $0 \leq j < i$ (as the identification of $(n-1)$-cells within a gmap can induce the identification of j-cells, $0 \leq j \leq n-2$). So, the association of homological equivalences with cells for the resulting cmap is done as for gmaps. Moreover, it is easy to see that the computation of the homological equivalence associated with the connected components of the cmap can be done according to the construction presented in the proof of theorem 4.11.

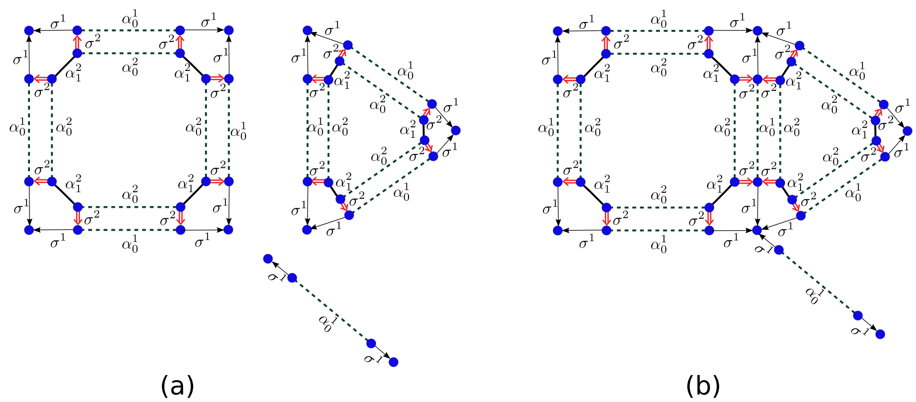


Figure 21: (a) a cmap: each cell is a gmap, σ^i link $i-$cells with $(i-1)$-cells. (b) the cmap resulting from some identifications.

Extension to incidence graphs

Incidence graphs [Ber99] and n-surfaces [DCB05] make it possible to represent "cellular" objects without multi-incidence. n-surfaces are equivalent to gmaps without multi-incidence [ADLL08], and when cells are quasi-manifolds, incidence graphs are equivalent to cmaps without multi-incidence [ALP15].

Under this assumption:

- a connected n-surface corresponds to a connected n-dimensional gmap without boundary and without multi-incidence. A cellular cone on the n-surface consists in adding a $(n+1)$-cell, incident to all n-cells. The result is an incidence graph which contains only one $(n+1)$-cell and its boundary;
- assume that an incidence graph is constructed by adding such cells (with their boundaries). It is possible to identify any i-cells (and their boundaries), for any i, if this does not induce multi-incidence and if the subgraphs corresponding to these i-cells and their boundaries are isomorphic.

The first operation corresponds to the gmap extension. An incidence graph made of isolated k-cells and their boundaries (for $0 \le k \le n$) corresponds to a cmap in which each connected component corresponds to a connected gmap containing one k-cell. The identification of isomorphic i-cells in an incidence graph corresponds to the identification of isomorphic cells in the corresponding cmap.

Moreover, it is possible also to directly associate a simplicial analog with an incidence graph, i.e. a semi-simplicial set structured into cells, so it is possible to directly associate a chain complex with an incidence graph[22]. So, all results described above for gmaps and cmaps can be directly applied for n-surfaces and for incidence graphs which cells are quasi-manifolds.

5 Conclusion and perspectives

The main results presented in this paper are:

1. For semi-simplicial sets: it has been shown that a short exact sequence can be associated with any identification (cf. lemma 3.6), and it is possible to compute a homological equivalence associated with the result of an identification operation by one application of the SES theorem (cf. theorem 3.7). The complexity of the computation of a homological equivalence resulting from the application of the cone or identification operations is also discussed; it is linear for the cone operation, but for the identification operation, the complexity depends also on the operations which have been applied previously; at last, note that theorem 3.7 generalizes in some sense (cf. Section 3.2.2) the approach of Boltcheva *et al* [BMLH10, BCMA+11] for the incremental computation of the homology of abstract simplicial complexes;

2. For gmaps, it is proposed to associate a homological equivalence not only with the gmap, but also with any of its cells. In this way, it is possible, for gmaps without self-bending, to compute a homological equivalence associated with:

   - the result of the extension operation, in linear time according to the size of the initial gmap and its associated homological equivalences: cf. theorem 4.10;
   - the result of the sewing operation, by one application of the SES theorem: cf. theorem 4.11. The construction of a homological equivalence characterizing the identification can be done in linear time according to (at most) the size of the homological equivalences associated with the gmap times the dimension of the gmap;

3. A discussion about the extension of these results for other simplicial and cellular structures (abstract simplicial complexes, simplicial sets, chains of maps, n-surfaces and incidence graphs), showing it is straightforward.

Future works will deal with the study of:

[22] Note that the simplicial analogs associated with an incidence graph and its equivalent cmap are isomorphic.

1. For cellular structures:
   - the identification operation involving self-bending of cells;
   - other construction operations: variants of the cone operation, contraction or removing cells [DL03], chamfering, etc;
2. For all structures:
   - the simplification of homological equivalences, mainly by applying elementary reduction (cf. Annex 6.2), or computation of vector fields [RS10];
   - the complexity related to these operations, and mainly simplifications, since it has to be taken into account for the other operations, and mainly when applying the SES theorem.

Acknowledgments:
The authors gratefully acknowledge Francis Sergeraert for many helpful discussions and encouragements. Many thanks also to Pol Vanhaecke et Claude Quitté.

References

[ADLL08] S. Alayrangues, X. Daragon, J.-O. Lachaud, and P. Lienhardt. Equivalence between closed connected n-g-maps without multi-incidence and n-surfaces. *J. Math. Imaging Vis.*, 32(1):1–22, 2008.

[ADLP15] S. Alayrangues, G. Damiand, P. Lienhardt, and S. Peltier. Homology of cellular structures allowing multi-incidence. *Discrete and Computational Geometry*, to appear, accepted January 2015.

[Ago76] M. K. Agoston. *Algebraic Topology, a first course.* Pure and applied mathematics. Marcel Dekker Ed., 1976.

[ALP15] S Alayrangues, P Lienhardt, and S Peltier. Conversion between chains of maps and chains of surfaces; application to the computation of incidence graphs homology. Research report, Université de Poitiers, 2015. https://hal.archives-ouvertes.fr/hal-01130543/file/RR-2014-11.pdf.

[Bas10] T. Basak. Combinatorial cell complexes and poincaré duality. *Geometriae Dedicata*, 147:357–387, 2010.

[BCMA+11] D. Boltcheva, D. Canino, S. Merino Aceituno, J.-C. Léon, L. De Floriani, and F. Hétroy. An iterative algorithm for homology computation on simplical shapes. *Computer-Aided Design*, 43(11):1457–1467, 2011.

[Ber99] G. Bertrand. New notions for discrete topology. In *Proc. of 8th Disc. Geom. for Computer Imagery, Marne-la-Vallée*, volume 1568 of *LNCS*, pages 218–228. Springer-Verlag, 1999.

[BMLH10] D. Boltcheva, S. Merino, J.-C. Léon, and F. Hétroy. Constructive Mayer-Vietoris Algorithm: Computing the Homology of Unions of Simplicial Complexes. Research Report RR-7471, INRIA, December 2010. http://hal.inria.fr/inria-00542717/PDF/RR-7471.pdf.

[Bou89] N. Bourbaki. *Algebra I: Chapters 1-3*. Elements of mathematics. Springer, 1989.

[BPA+10] T. Bellet, M. Poudret, A. Arnould, P. Le Gall, and L. Fuchs. Designing a Topological Modeler Kernel: A Rule-Based Approach. In *Shape Modeling International (SMI'10)*, Aix-en-Provence, France, 2010.

[Bri93] E. Brisson. Representing geometric structures in d dimensions: Topology and order. *Discrete & Computational Geometry*, 9:387–426, 1993.

[BSP+05] S. Brandel, S. Schneider, M. Perrin, N. Guiard, J.-F. Rainaud, P. Lienhardt, and Y. Bertrand. Automatic building of structured geological models. *Journal of Computing and Information Science in Ingeneering, Volume 5, numéro 2*, 2005.

[DCB05] X. Daragon, M. Couprie, and G. Bertrand. Discrete surfaces and frontier orders. *Journal of Mathematical Imaging and Vision*, 23:379–399, 2005.

[DD09] A. Dupas and G. Damiand. Region merging with topological control. *Discrete Applied Mathematics*, 157(16):3435–3446, August 2009.

[DHSW03] J.-G. Dumas, F. Heckenbach, B. D. Saunders, and V Welker. Computing simplicial homology based on efficient smith normal form algorithms. In *Algebra, Geometry, and Software Systems*, pages 177–206, 2003.

[DKMW11] P. Dlotko, T. Kaczynski, M. Mrozek, and T. Wanner. Coreduction homology algorithm for regular cw-complexes. *Discrete & Computational Geometry*, 46(2):361–388, 2011.

[DL03] G. Damiand and P. Lienhardt. Removal and contraction for n-dimensional generalized maps. In *Proc. of 11th International Conference on Discrete Geometry for Computer Imagery (DGCI)*, volume 2886 of *Lecture Notes in Computer Science*, pages 408–419, Naples, Italy, November 2003. Springer Berlin/Heidelberg.

[DL14] G. Damiand and P. Lienhardt. *Combinatorial Maps: Efficient Data Structures for Computer Graphics and Image Processing.* A K Peters/CRC Press, September 2014.

[DSV01] J. G. Dumas, B.D. Saunders, and G. Villard. On efficient sparse integer matrix smith normal form computations. *J. of Symbolic Computation*, 2001.

[EH10] Herbert Edelsbrunner and John Harer. *Computational Topology - an Introduction.* American Mathematical Society, 2010.

[EL95] H. Elter and P. Lienhardt. Cellular complexes as structured semi-simplicial sets. *International Journal of Shape Modeling*, 1(2):191–217, 1995.

[GDJMR09] R. González-Díaz, M. J. Jiménez, B. Medrano, and P. Real. Chain homotopies for object topological representations. *Discrete Applied Mathematics*, 157(3):490–499, 2009.

[Gie96] M. Giesbrecht. Probabilistic computation of the smith normal form of a sparse integer matrix. In *Proceedings of the Second Int. Symp. on Algorithmic Number Theory, LNCS*, volume 1122, pages 173–186. Springer-Verlag, 1996.

[KMM04] T. Kaczynski, K. Mischaikow, and M. Mrozek. *Computational Homology.* Applied Mathematical Sciences. Springer, 2004.

[Lie94] P. Lienhardt. N-dimensional generalized combinatorial maps and cellular quasi-manifolds. *Int. J. on Comput. Geom. & App.*, 4(3):275–324, 1994.

[LL95] V. Lang and P. Lienhardt. Geometric modeling with simplicial sets. In T.L. Kunii S.Y. Shin, editor, *Computer Graphics and Applications*, Pacific Graphics, pages 475–494, Seoul, Corea, 1995. World Scientific Publishing.

[May67] J. P. May. *Simplicial Objects in Algebraic Topology.* Van Nostrand, 1967.

[MB09] M. Mrozek and B. Batko. Coreduction homology algorithm. *Discrete and Computational Geometry*, 41(1):96–118, 2009.

[PAFL06] S. Peltier, S. Alayrangues, L. Fuchs, and J.-O. Lachaud. Computation of homology groups and generators. *Comput. & Graph.*, 30:62–69, feb. 2006.

[PFL09] S. Peltier, L. Fuchs, and P. Lienhardt. Simploidals sets: Definitions, operations and comparison with simplicial sets. *Discrete App. Math.*, 157:542–557, feb. 2009.

[RS06] J. Rubio and F. Sergeraert. Constructive homological algebra and applications. Genova, Italy, August, 28 - September, 02 2006. Genova Summer School on Mathematics, Algorithms, Proofs. http://arxiv.org/abs/1208.3816.

[RS10] A. Romero and F. Sergeraert. Discrete vector fields and fundamental algebraic topology. *CoRR*, abs/1005.5685, 2010. http://arxiv.org/abs/1005.5685.

[SDdJ15] C. Solnon, G. Damiand, C. de la Higuera, and J.-C. Janodet. On the complexity of submap isomorphism and maximum common submap problems. *Pattern Recognition (PR)*, 48(2):302–316, February 2015.

[Sto96] A. Storjohann. Near optimal algorithms for computing smith normal forms of integer matrices. In Y. N. Lakshman, editor, *Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation*, pages 267–274. ACM, 1996.

6 Annex

6.1 Basic algebraic notions

Definition 6.1 *Module over a commutative ring [Bou89] (II.1.1 p191)*

Given a commutative ring A, a module over A or A-module is a set E with an algebraic structure by giving :

1. *a commutative group law on E (written additively in what follows);*
2. *a law of action $(\alpha, x) \in A \times E \mapsto \alpha T x \in E$, which satisfies the following axioms.*
   - *(a)* $\alpha T(x + y) = (\alpha T x) + (\alpha T y), \forall \alpha \in A, x \in E, y \in E,$
   - *(b)* $(\alpha + \beta)T x = (\alpha T x) + (\beta T x), \forall \alpha \in A, \beta \in A, x \in E,$
   - *(c)* $\alpha T(\beta T x) = (\alpha\beta)T x, \forall \alpha \in A, \beta \in A, x \in E,$
   - *(d)* $1T x = x, \forall x \in E.$

Definition 6.2 *A-morphism [Bou89] (II.1.2 p194)*

Let E and F be two modules with respect to the same ring A. An A-morphism of E into F is any mapping $\delta : E \mapsto F$ such that:

$$(x + y)\delta = x\delta + y\delta \text{ for } x \in E \text{ and } y \in E$$

$$(\lambda T x)\delta = \lambda T(x\delta) \text{ for } \lambda \in A \text{ and } x \in E$$

Definition 6.3 *Graded commutative group [Bou89] (II.11.1 p363)*
Given a commutative group G and a set Δ, a graduation of type Δ on G is a family $(G_{\lambda,\lambda\in\Delta})$ of subgroups of G of which G is a direct sum. The set G with a structure defined by its group law and its graduation is called a graded commutative group of type Δ.

Definition 6.4 *Graded ring [Bou89] (II.11.2 p364)*
Given a ring A, and a graduation (A_λ) of the type Δ on the additive group A, this graduation is said to be compatible with a ring structure on A if

$$A_\lambda A_\mu \subset A_{\lambda+\mu}, \forall\lambda,\mu\in\Delta$$

where $A_\lambda A_\mu$ denotes the set $\{a_\lambda a_\mu, a_\lambda\in A_\lambda, a_\mu\in A_\mu\}$.
This ring A with this graduation is called a graded ring of type Δ.

Definition 6.5 *Graded module [Bou89] (II.11.2 p365)*
Let A be a graded commutative ring of type Δ, (A_λ) its graduation and M a A-module; a graduation (M_λ) of type Δ on the additive group M is compatible with the A-module structure on M if

$$A_\lambda M_\mu \subset M_{\lambda+\mu}, \forall\lambda,\mu\in\Delta$$

where $A_\lambda M_\mu$ denotes the set $\{a_\lambda m_\mu, a_\lambda\in A_\lambda, m_\mu\in M_\mu\}$.
The module M with this graduation is then called a graded module of type Δ over the graded ring A.

Definition 6.6 *Graded module morphism [Bou89] (II.11.2 p366)*
Let A, A' be two graded rings of the same type Δ and (A_λ), (A'_λ) their respective graduations. A ring morphism $\rho : A \mapsto A'$ is called graded if $A_\lambda\rho \subset A'_\lambda, \forall\lambda\in\Delta$.
Let M, M' be two graded modules of type Δ over a graded commutative ring of type Δ. Let $\delta : M \mapsto M'$ be an A-morphism and μ an element of Δ, δ is called graded of degree μ if $M_\lambda\delta \subset M'_{\lambda+\mu}, \forall\lambda\in\Delta$.

6.2 Elementary reduction

An *elementary reduction* is defined as follows.

Definition 6.7 *Let (C^0, ∂^0) be a chain complex. Let x and y be two elements of C^0 such that x appears in the boundary of y with a coefficient $\epsilon = \pm 1$. The homotopy operator h^{01} is defined by:*

- $xh^{01} = \epsilon y$
- $\forall z \neq x, z\ h^{01} = 0$

Then:

- C^1 *is a graded module isomorphic to* C^0-$\{x, y\}$ *by a one-to-one mapping* ϕ.
- $f^{01} : C^0 \longrightarrow C^1$ *is such that:*
  - $\forall z \neq x, y,\ zf^{01} = z\phi$
  - $yf^{01} = 0$
  - $xf^{01} = (-\epsilon y\partial^0 + x)\phi$
- $g^{01} : C^1 \longrightarrow C^0$ *is such that:*
  $\forall z,\ zg^{01} = z\phi^{-1} - z\phi^{-1}\partial^0 h^{01}$
- ∂^1 *is such that:*
  $\forall z,\ z\partial^1 = z\phi^{-1}\partial^0 f^{01}$

This elementary reduction is denoted $x \to \epsilon y$.

Property 6.8 *An elementary reduction is a reduction.*

Proof *We can easily state that:*

- $h^{01}h^{01} = 0$ *since:*
  - $xh^{01}h^{01} = \epsilon yh^{01} = 0$
  - $\forall z \neq x,\ zh^{01} = 0$
- C^1 *is a graded module since* C^0 *is one.*
- f^{01} *is a graded module morphism since*
  - $\forall z \neq x, y,\ zf^{01} = z\phi$
  - $yf^{01} = 0$
  - $xf^{01} = (-\epsilon y\partial^0 + x)\phi$, *where* $-\epsilon y\partial^0$ *and* x *have the same dimension, and* ϕ *preserves dimension.*
- g^{01} *is a graded module morphism, since* $g^{01} = \phi^{-1} - \phi^{-1}\partial^0 h^{01}$ *is a composition of graded module morphisms, where* ϕ^{-1} *preserves dimension, as* $\partial^0 h^{01}$.
- ∂^1 *is a graded module morphism of degree* -1 *since* $\partial^1 = \phi^{-1}\partial^0 f^{01}$ *is a composition of graded module morphisms, where* ϕ^{-1} *and* f^{01} *preserve dimension, and the degree of* ∂^0 *is* -1.
- $h^{01}f^{01} = 0$ *since*
  - $xh^{01}f^{01} = \epsilon yf^{01} = 0$
  - $\forall z \neq x,\ zh^{01} = 0$
- $g^{01}h^{01} = 0$ *since* $\forall z, zg^{01}h^{01} = (z\phi^{-1} - z\phi^{-1}\partial^0 h^{01})h^{01} = z\phi^{-1}h^{01} = 0$ *because* $z\phi^{-1} \neq x, y$.
- $\partial^1\partial^1 = \phi^{-1}\partial^0 f^{01}\phi^{-1}\partial^0 f^{01} = 0$, *since:*

- let z be such that $z\phi^{-1}\partial^0 = \sum_j \alpha'_j z'_j$ and $\forall j, x, y \neq z'_j$. Then
  $z\phi^{-1}\partial^0 f^{01}\phi^{-1}\partial^0 f^{01}$
  $= (\sum_j \alpha'_j z'_j \phi\phi^{-1})\partial^0 f^{01}$
  $= z\phi^{-1}\partial^0\partial^0 f^{01}$
  $= 0$ (since $z'_j f^{01} = z'_j\phi, \forall j$)
- Let z be such that $y \in z\phi^{-1}\partial^0$, we obtain the same result as before since $yf^{01} = 0$.
- let z be such that $z\phi^{-1}\partial^0 = \sum_j \alpha'_j z'_j + \alpha x$ and $\forall j, x, y \neq z'_j$. Then $z\phi^{-1}\partial^0 f^{01}\phi^{-1}\partial^0 f^{01}$
  $= (\sum_j \alpha'_j z'_j \phi\phi^{-1})\partial^0 f^{01} + \alpha(-\epsilon y\partial^0 + x)\phi\phi^{-1}\partial^0 f^{01}$
  $= (\sum_j \alpha'_j z'_j + \alpha x)\partial^0 f^{01} - \alpha\epsilon y\partial^0\partial^0 f^{01}$
  $= z\phi^{-1}\partial^0\partial^0 f^{01}$
  $= 0$

- $f^{01}\partial^1 = \partial^0 f^{01}$, i.e. f^{01} is a chain complex morphism, since
  - let $z \neq x, y$; $zf^{01}\partial^1 = z\phi\phi^{-1}\partial^0 f^{01} = z\partial^0 f^{01}$
  - $yf^{01}\partial^1 = 0$;
    $y\partial^0 f^{01}$
    $= ((y\partial^0 - \epsilon x) + \epsilon x)f^{01}$
    $= ((y\partial^0 - \epsilon x) + \epsilon(-\epsilon y\partial^0 + x))\phi$
    $= (y\partial^0 - \epsilon^2 y\partial^0 - \epsilon x + \epsilon x)\phi$
    $= 0$
  - $xf^{01}\partial^1$
    $= -\epsilon y\partial^0\phi\partial^1 + x\phi\partial^1$
    $= -\epsilon y\partial^0\phi\phi^{-1}\partial^0 f^{01} + x\phi\phi^{-1}\partial^0 f^{01}$
    $= x\partial^0 f^{01}$
- $g^{01}f^{01} = id_{C^1}$ since
  $\forall z, (z\phi^{-1} - z\phi^{-1}\partial^0 h^{01})f^{01}$
  $= z\phi^{-1}\phi - z\phi^{-1}\partial^0 h^{01} f^{01}$
  $= z$
- $f^{01}g^{01} + h^{01}\partial^0 + \partial^0 h^{01} = id_{C^0}$ since
  - $\forall z \neq x, y,\ z\phi\phi^{-1} - z\phi\phi^{-1}\partial^0 h^{01} + zh^{01}\partial^0 + z\partial^0 h^{01} = z$
  - $yf^{01}g^{01} + yh^{01}\partial^0 + y\partial^0 h^{01} = (\sum_i \alpha_i z_i + \epsilon x)h^{01} = \epsilon\epsilon y = y$
  - $xf^{01}g^{01} + xh^{01}\partial^0 + x\partial^0 h^{01}$
    $= ((-\epsilon y\partial^0 + x)\phi)(\phi^{-1} - \phi^{-1}\partial^0 h^{01}) + xh^{01}\partial^0 + x\partial^0 h^{01}$
    $= -\epsilon y\partial^0 + \epsilon y\partial^0\partial^0 h^{01} + x - x\partial^0 h^{01} + xh^{01}\partial^0 + x\partial^0 h^{01}$
    $= -\epsilon y\partial^0 + x + \epsilon y\partial^0$
    $= x$
- g^{01} is a chain complex morphism since
  $\partial^1 g^{01} = \phi^{-1}\partial^0 f^{01} g^{01}$
  $= \phi^{-1}\partial^0(id_{C^0} - h^{01}\partial^0 - \partial^0 h^{01})$
  $= \phi^{-1}\partial^0 - \phi^{-1}\partial^0 h^{01}\partial^0 - \phi^{-1}\partial^0\partial^0 h^{01}$

$$= (\phi^{-1} - \phi^{-1}\partial^0 h^{01})\partial^0$$
$$= g^{01}\partial^0$$

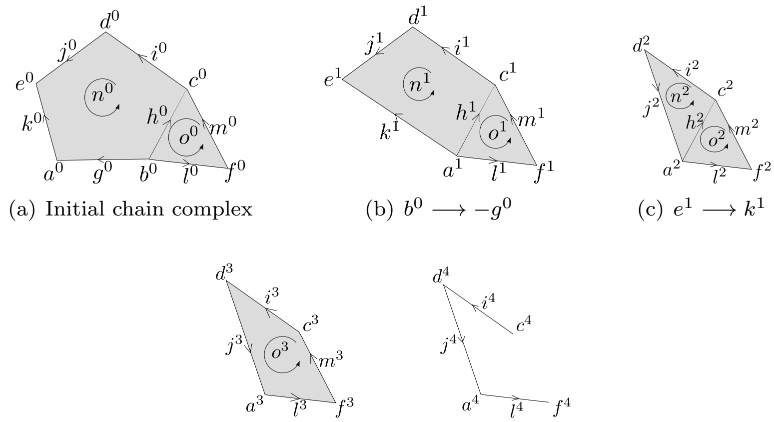
□

Example 7 *Sequence of elementary reductions*

A sequence of elementary reductions:

- $\rho^{01} = ((C^0, \partial^0), (C^1, \partial^1), h^{01}, f^{01}, g^{01})$,
- $\rho^{12} = ((C^1, \partial^1), (C^2, \partial^2), h^{12}, f^{12}, g^{12})$,
- $\rho^{23} = ((C^2, \partial^2), (C^3, \partial^3), h^{23}, f^{23}, g^{23})$,
- $\rho^{34} = ((C^3, \partial^3), (C^4, \partial^4), h^{34}, f^{34}, g^{34})$

is represented on Fig. 22 (graphical representation) and Fig. 23 (formal definition).



(a) Initial chain complex (b) $b^0 \longrightarrow -g^0$ (c) $e^1 \longrightarrow k^1$ (d) $h^2 \longrightarrow n^2$ (e) $m^3 \longrightarrow o^3$

Figure 22: Sequence of elementary reductions (remember that $x \to \epsilon y$ denotes the elementary reduction characterized by $xh = \epsilon y$). The *generators* of the different chain groups are graphically represented. A i-dimensional generator is represented by a i-dimensional cell. A 0-generator appears in the boundary of a 1-generator with 1 as coefficient (resp. -1) if the corresponding vertex is the extremity (resp. origin) of the corresponding edge. Each face is assumed to be oriented counterclockwise: a 1-generator appears in the boundary of a 2-generator with 1 as coefficient (resp. -1) if the orientation of the associated edge corresponds to (resp. is inverse to) the orientation of the associated face. The exponent associated with each cell allows to distinguish the different chain complexes.

C^0	a^0	$\mathbf{b^0}$	c^0	d^0	e^0	f^0	g^0	h^0	i^0	j^0	k^0	l^0	m^0	n^0	o^0
∂^0	0	0	0	0	0	0	a^0-b^0	c^0-b^0	d^0-c^0	e^0-d^0	e^0-a^0	f^0-b^0	c^0-f^0	$-g^0+h^0+i^0+j^0-k^0$	$l^0+m^0-h^0$
$\mathbf{h^{01}}$	0	$-\mathbf{g^0}$	0	0	0	0	0	0	0	0	0	0	0	0	0
f^{01}	a^1	a^1	c^1	d^1	e^1	f^1	0	h^1	i^1	j^1	k^1	l^1	m^1	n^1	o^1
C^1	a^1		c^1	d^1	$\mathbf{e^1}$	f^1		h^1	i^1	j^1	k^1	l^1	m^1	n^1	o^1
∂^1	0		0	0	0	0		c^1-a^1	d^1-c^1	e^1-d^1	e^1-a^1	f^1-a^1	c^1-f^1	$h^1+i^1+j^1-k^1$	$l^1+m^1-h^1$
g^{01}	a^0		c^0	d^0	e^0	f^0		h^0-g^0	i^0	j^0	k^0	l^0-g^0	m^0	n^0	o^0
$\mathbf{h^{12}}$	0		0	0	$\mathbf{k^1}$	0		0	0	0	0	0	0	0	0
f^{12}	a^2		c^2	d^2	a^2	f^2		h^2	i^2	j^2	0	l^2	m^2	n^2	o^2
C^2	a^2		c^2	d^2		f^2		$\mathbf{h^2}$	i^2	j^2		l^2	m^2	n^2	o^2
∂^2	0		0	0		0		c^2-a^2	d^2-c^2	a^2-d^2		f^2-a^2	c^2-f^2	$h^2+i^2+j^2$	$l^2+m^2-h^2$
g^{12}	a^1		c^1	d^1		f^1		h^1	i^1	j^1-k^1		l^1	m^1	n^1	o^1
$\mathbf{h^{23}}$	0		0	0		0		$\mathbf{n^2}$	0	0		0	0	0	0
f^{23}	a^3		c^3	d^3		f^3		$-i^3-j^3$	i^3	j^3		l^3	m^3	0	o^3
C^3	a^3		c^3	d^3		f^3			i^3	j^3		l^3	$\mathbf{m^3}$		o^3
∂^3	0		0	0		0			d^3-c^3	a^3-d^3		f^3-a^3	c^3-f^3		$i^3+j^3+l^3+m^3$
g^{23}	a^2		c^2	d^2		f^2			i^2	j^2		l^2	m^2		n^2+o^2
$\mathbf{h^{34}}$	0		0	0		0			0	0		0	$\mathbf{o^3}$		0
f^{34}	a^4		c^4	d^4		f^4			i^4	j^4		l^4	$-i^4-j^4-l^4$		0
C^4	a^4		c^4	d^4		f^4			i^4	j^4		l^4			
∂^4	0		0	0		0			d^4-c^4	a^4-d^4		f^4-a^4			
g^{34}	a^3		c^3	d^3		f^3			i^3	j^3		l^3			

Figure 23: Formal definition of the sequence of elementary reductions depicted on Fig. 22. For instance, the row starting with C^0 contains its generators, from dimension 0 to dimension 2. The following rows, until the row starting with C^1, contains the definitions of the mappings defined on C^0, e.g. $b^0 h^{01} = -g^0$, since $-g^0$ is at the intersection of the row corresponding to h^{01} and of the column corresponding to b^0 in the row associated with C^0.

The properties of any reduction (C, C', f, g, h) can be explained on an elementary reduction example, for instance, on ρ^{01} (see Figure 22(b)).

The homotopy operator, h, associates the elements to be suppressed through the reduction process (for instance here, h^{01} links b^0 and $-g^0$). Roughly speaking, it generalizes the notion of collapse [KMM04].

f and g are inverse to each other on the remaining elements. The element which is "shrinked" has a null image by f (for instance, $g^0 f^{01} = 0$). The element associated with it in the reduction is sent on the remaining part of the boundary of the "shrinked" element (for instance, $b^0 f^{01} = a^1$). This explains why $gf = id$ and $id - fg = h\partial + \partial h$ since $id - fg$ is the difference in the initial object between an element and the elements corresponding to the elements on which it is sent. For instance:

- *for h^0:*
  - $h^0(id - f^{01}g^{01}) = h^0 - (h^0 - g^0) = g^0$,
  - $h^0 h^{01} \partial^0 = 0$
  - $h^0 \partial^0 h^{01} = (c^0 - b^0)h^{01} = 0 - (-g^0)$;
- *for b^0:*
  - $b^0(id - f^{01}g^{01}) = b^0 - a^0$,
  - $b^0 h^{01} \partial^0 = -g^0 \partial^0 = -(a^0 - b^0)$,
  - $b^0 \partial^0 h^{01} = 0$.

More generally, we can see in example 8 that:

- $h^0(id - f^{03}g^{03}) = h^0 - (-i^3 - j^3)g^{03} = h^0 - (-i^0 - (j^0 - k^0)) = h^0 + i^0 + j^0 - k^0$,
- $h^0 h^{03} \partial^0 = n^0 \partial^0 = -g^0 + h^0 + i^0 + j^0 - k^0$,
- $h^0 \partial^0 h^{03} = (c^0 - b^0)h^{03} = 0 - (-g^0)$.

The three last properties can be explained in the following way:

- *$hf = 0$, since for each element x, either $xh = 0$, or $xh \neq 0$ and this "shrinked" element has no corresponding element in the resulting chain complex.*
- *$gh = 0$, since g associates with a remaining element an element of the initial chain complex, which is either an element which remains (on which $h = 0$) or a "shrinked" one (on which $h = 0$ also). For instance, $j^2 g^{12} = j^1 - k^1$, intuitively j^1 has been extended on $-k^1$ through e^1 (note also that e^1 has no preimage in C^2).*
- *$hh = 0$, since for each element x, either $xh = 0$, or $xh \neq 0$ is a "shrinked" element which cannot be used in order to "shrink" another element.*

□

Example 8 *Composition of elementary reductions*

Let $\rho^{02} = ((C^0, \partial^0), (C^2, \partial^2), h^{02}, f^{02}, g^{02})$ (resp. $\rho^{03} = ((C^0, \partial^0), (C^3, \partial^3), h^{03}, f^{03}, g^{03})$) be the composition of ρ^{01} and ρ^{12} (resp. ρ^{02} and ρ^{23}) of example 7. ρ^{02} and ρ^{03} are described in Fig. 24.

C^0	a^0	b^0	c^0	d^0	e^0	f^0	g^0	h^0	i^0	j^0	k^0	l^0	m^0	n^0	o^0
∂^0	0	0	0	0	0	0	a^0-b^0	c^0-b^0	d^0-c^0	e^0-d^0	e^0-a^0	f^0-b^0	c^0-f^0	$-g^0+h^0+i^0+j^0-k^0$	$l^0+m^0-h^0$
h^{02}	0	$-g^0$	0	0	k^0	0	0	0	0	0	0	0	0	0	0
f^{02}	a^2	a^2	c^2	d^2	a^2	f^2	0	h^2	i^2	j^2	0	l^2	m^2	n^2	o^2
C^2	a^2		c^2	d^2		f^2		h^2	i^2	j^2		l^2	m^2	n^2	o^2
∂^2	0		0	0		0		c^2-a^2	d^2-c^2	a^2-d^2		f^2-a^2	c^2-f^2	$h^2+i^2+j^2$	$l^2+m^2-h^2$
g^{02}	a^0		c^0	d^0		f^0		h^0-g^0	i^0	j^0-k^0		l^0-g^0	m^0	n^0	o^0
h^{03}	0	$-g^0$	0	0	k^0	0	0	n^0	0	0	0	0	0	0	0
f^{03}	a^3	a^3	c^3	d^3	a^3	f^3	0	$-i^3-j^3$	i^3	j^3	0	l^3	m^3	0	o^3
C^3	a^3		c^3	d^3		f^3			i^3	j^3		l^3	m^3		o^3
∂^3	0		0	0		0			d^3-c^3	a^3-d^3		f^3-a^3	c^3-f^3		$i^3+j^3+l^3+m^3$
g^{03}	a^0		c^0	d^0		f^0			i^0	j^0-k^0		l^0-g^0	m^0		n^0+o^0

Figure 24: Composition of elementary reductions depicted in Fig. 22.

□

Remark 2 *Not any reduction can be decomposed into elementary reductions. For instance, let us consider the chain complex:*

$$\cdots 0 \longrightarrow C_i = \mathbb{Z} \times \mathbb{Z} \xrightarrow{\partial^i} C_{i-1} = \mathbb{Z} \times \mathbb{Z} \longrightarrow 0 \cdots$$

with $\forall j > i$ or $j < i-1$, $C_j = 0$, and

$$\partial^i = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$$

Note that C_i and C_{i-1} have the same generators $(1,0)$ and $(0,1)$.
Let the null complex be

$$\cdots 0 \longrightarrow D_i = 0 \xrightarrow{0} D_{i-1} = 0 \longrightarrow 0 \cdots$$

with $\forall j > i$ or $j < i-1$, $D_j = 0$.
We can easily check that a reduction of (C, ∂) onto $(D, 0)$ is defined by:

- *The homotopy operator h such that*

$$h_{i-1} = \begin{pmatrix} 3/2 & -2 \\ -1 & 3/2 \end{pmatrix} : C_{i-1} \longrightarrow C_i$$

and $\forall j \neq i-1, h_j$ *is null.*

- $f : C \longrightarrow D$ *and* $g : D \longrightarrow C$*, both null.*

The generators $(0,1)$ *and* $(1,0)$ *appear with a coefficient different from* ± 1 *in the boundary of each generator. Hence there exists no elementary reduction.*

6.3 Consistence of Definition 3.2

Definition 3.2 (cone of a morphism) is consistent, since:

- obviously C^ϕ is a graded module and ∂_i^ϕ is a graded module morphism of degree -1
- $\forall i, \partial_i^\phi \partial_{i-1}^\phi = \begin{pmatrix} -\partial_{i-1}^0 & \phi_{i-1} \\ 0 & \partial_i^1 \end{pmatrix} \begin{pmatrix} -\partial_{i-2}^0 & \phi_{i-2} \\ 0 & \partial_{i-1}^1 \end{pmatrix}$
  $= \begin{pmatrix} \partial_{i-1}^0 \partial_{i-2}^0 & -\partial_{i-1}^0 \phi_{i-2} + \phi_{i-1} \partial_{i-1}^1 \\ 0 & \partial_i^1 \partial_{i-1}^1 \end{pmatrix}$
  $= 0,$
  since ∂^0 and ∂^1 are boundary operators and ϕ is a chain complex morphism, i.e. $\forall i, \phi_i \partial_i^1 = \partial_i^0 \phi_{i-1}$.

6.4 Proof of Easy and Basic Pertubation Lemmas

Proof *of Easy Perturbation Lemma.*

h is, by definition, a homotopy operator. f and g are graded module morphisms. For the same reason, $gf = id$*,* $hf = gh = hh = 0$*.*

- δ^0 *is a perturbation of* ∂^0 *since* $\partial^0\delta^0 + \delta^0\partial^0 + (\delta^0)^2 = (\partial^0 f)\delta^1 g + f\delta^1(g\partial^0) + f\delta^1(gf)\delta^1 g = f(\partial^1\delta^1 + \delta^1\partial^1 + \delta^1\delta^1)g = 0.$
- f *is a chain complex morphism since* $(\partial^0 + \delta^0)f = f\partial^1 + f\delta^1(gf) = f(\partial^1 + \delta^1)$.
- g *is a chain complex morphism since* $g(\partial^0 + \delta^0) = \partial^1 g + (gf)\delta^1 g = (\partial^1 + \delta^1)g$.
- $fg + h(\partial^0 + \delta^0) + (\partial^0 + \delta^0)h = (fg + h\partial^0 + \partial^0 h) + (hf)\delta^1 g + f\delta^1(gh) = id$

□

Proof *of Basic Perturbation Lemma.*

Note that Φ *and* Ψ *satisfy the following properties:*

- $h\Phi = \Psi h$
- $\Phi\delta^0 = \delta^0\Psi$
- $\Phi = id - \Phi\delta^0 h = id - \delta^0 h\Phi = id - \delta^0\Psi h$

- $\Psi = id - \Psi h\delta^0 = id - h\delta^0\Psi = id - h\Phi\delta^0$.

These properties of are obviously deduced from the definitions of Φ *and* Ψ*, as* $\Phi\delta^0 h = \delta^0 h\Phi = \delta^0\Psi h = id - \Phi$ *and* $\Psi h\delta^0 = h\delta^0\Psi = h\Phi\delta^0 = id - \Psi$*.*

- δ^1 *is a perturbation of* ∂^1 *since* $\partial^1\delta^1 + \delta^1\partial^1 + \delta^1\delta^1 = (\partial^1 g)\delta^0\Psi f + g\Phi\delta^0(f\partial^1) + g\Phi\delta^0(fg)\Phi\delta^0 f = g(\partial^0\delta^0\Psi + \Phi\delta^0\partial^0 + \Phi\delta^0(\Phi\delta^0) - \Phi\delta^0\partial^0(h\Phi\delta^0) - (\Phi\delta^0 h)\partial^0(\Phi\delta^0))f = g(\partial^0\delta^0\Psi + \Phi\delta^0\partial^0 + \Phi\delta^0\delta^0\Psi - \Phi\delta^0\partial^0(id - \Psi) - (id - \Phi)\partial^0\delta^0\Psi)f = g\Phi(\partial^0\delta^0 + \delta^0\partial^0 + \delta^0\delta^0)\Psi f = 0$
- Φ *and* Ψ *obviously being graded module morphisms,* f^δ *and* g^δ *are graded module morphisms and* h^δ *is a homotopy operator*
- $\Phi\Psi = (id - \Phi\delta^0 h)(id - h\delta^0\Psi) = id - (id - \Psi) - (id - \Phi) + \Phi\delta^0(hh)\delta^0\Psi = \Phi + \Psi - id$.
- $\Psi\Phi = (id - \Psi h\delta^0)(id - \delta^0 h\Phi) = id - (id - \Phi) - (id - \Psi) + \Psi h\delta^0\delta^0 h\Phi = \Phi + \Psi - id + \Psi h\delta^0\delta^0 h\Phi$
- $\Phi f = f - \Phi\delta^0(hf) = f$
- $g\Psi = g - (gh)\delta^0\Psi = g$
- $h\Psi = \Phi h = h$
- $g^\delta f^\delta = g\Phi\Psi f = g(\Phi f) + (g\Psi)f - (gf) = id + id - id = id$
- $h^\delta f^\delta = h\Phi\Psi f = h(\Phi f) + (h\Psi)f - (hf) = 0$
- $g^\delta h^\delta = g\Phi\Psi h = g(\Phi h) + (g\Psi)h - (gh) = 0$
- $h^\delta h^\delta = h\Phi\Psi h = h(\Phi h) + (h\Psi)h - (hh) = 0$
- f^δ *is a chain complex morphism since* $\Psi f(\partial^1 + \delta^1) = \Psi(f\partial^1) + \Psi(fg)\delta^0\Psi f = \Psi\partial^0 f + \Psi\delta^0\Psi f - \Psi\partial^0(h\delta^0\Psi)f - \Psi h\partial^0\delta^0\Psi f = \Psi\partial^0 f + \Psi\delta^0(\Psi f) - \Psi\partial^0 f + \Psi\partial^0(\Psi f) - \Psi h\partial^0\delta^0(\Psi f) = ((id - \Psi h\delta^0)\delta^0 + (id - \Psi h\delta^0)\partial^0 - \Psi h\partial^0\delta^0)\Psi f = ((\partial^0 + \delta^0) - \Psi h(\delta^0\delta^0 + \delta^0\partial^0 + \partial^0\delta^0))f^\delta$
- g^δ *is a chain complex morphism since* $(\partial^1 + \delta^1)g\Phi = (\partial^1 g)\Phi + g\Phi\delta^0(fg)\Phi = g\partial^0\Phi + g\Phi\delta^0\Phi - g(\Phi\delta^0 h)\partial^0\Phi - g\Phi\delta^0\partial^0 h\Phi = g\partial^0\Phi + (g\Phi)\delta^0\Phi - g\partial^0\Phi + (g\Phi)\partial^0\Phi - (g\Phi)\delta^0\partial^0 h\Phi = g\Phi(\delta^0(id - \delta^0 h\Phi) + \partial^0(id - \delta^0 h\Phi) - \delta^0\partial^0 h\Phi) = g\Phi((\partial^0 + \delta^0) - (\delta^0\delta^0 + \partial^0\delta^0 + \delta^0\partial^0)h\Phi)$
- $f^\delta g^\delta + h^\delta(\partial^0 + \delta^0) + (\partial^0 + \delta^0)h^\delta = id$ *since* $\Psi(fg)\Phi + \Psi h\partial^0 + (\Psi h\delta^0) + \partial^0 h\Phi + (\delta^0 h\Phi) = (\Psi\Phi) - \Psi\partial^0 h\Phi - \Psi h\partial^0\Phi + \Psi h\partial^0 + (id - \Psi) + \partial^0 h\Phi + (id - \Phi) = (\Phi + \Psi - id + \Psi h\delta^0\delta^0 h\Phi) - \Psi\partial^0 h\Phi - \Psi h\partial^0\Phi + \Psi h\partial^0 + (id - \Psi) + \partial^0 h\Phi + (id - \Phi) = id + \Psi h\delta^0\delta^0 h\Phi - \Psi\partial^0 h\Phi - \Psi h\partial^0\Phi + \Psi h\partial^0 + \partial^0 h\Phi$

  Note that $\Psi h\delta^0\delta^0 h\Phi - \Psi\partial^0 h\Phi - \Psi h\partial^0\Phi + \Psi h\partial^0 + \partial^0 h\Phi = 0$ *since* $\Psi h\delta^0\delta^0 h\Phi - \Psi\partial^0 h\Phi - \Psi h\partial^0\Phi + \Psi h\partial^0 + \partial^0 h\Phi = \Psi h\delta^0\delta^0 h\Phi - (id - \Psi h\delta^0)\partial^0 h\Phi - \Psi h\partial^0(id - \delta^0 h\Phi) + \Psi h\partial^0 + \partial^0 h\Phi = \Psi h(\delta^0\delta^0 + \delta^0\partial^0 + \partial^0\delta^0)h\Phi$

□

$$\begin{tikzcd}
(C^0,\partial^0) \arrow[r, shift left, "i"] & (C^1,\partial^1) \arrow[l, shift left, "r"] \arrow[r, shift left, "j"] & (C^2,\partial^2) \arrow[l, shift left, "s"] \arrow[d, shift left, "g^i"] \\
(C^{B0},\partial^{B0}) \arrow[u, shift left, "f^0"] \arrow[loop left, "h^0"] \arrow[loop right, "h^{S0}"] \arrow[r, "i^B"] \arrow[d, shift right, "f^{S0}"'] & (C^{B1},\partial^{B1}) \arrow[u, shift left, "f^1"] \arrow[loop left, "h^1"] \arrow[loop right, "h^{S1}"] \arrow[d, shift right, "f^{S1}"'] & Cone(i) \arrow[u, shift left, "f^i"] \arrow[loop left, "h^i"] \arrow[d, shift left, "g^B"] \\
(C^{S0},\partial^{S0}) \arrow[r, "i^S"] \arrow[u, shift right, "g^{S0}"'] & (C^{S1},\partial^{S1}) \arrow[u, shift right, "g^{S1}"'] & Cone(i^B) \arrow[u, shift left, "f^B"] \arrow[loop left, "h^B"] \arrow[loop right, "h^2"] \arrow[d, shift right, "f^2"'] \\
& & Cone(i^S) \arrow[u, shift right, "g^2"']
\end{tikzcd}$$

Figure 25: Effective short exact sequence theorem.

6.5 Proof of SES Theorem

(see Figure 25)

Let

- $\rho^0 = ((C^{B0},\partial^{B0}),(C^0,\partial^0),h^0,f^0,g^0)$
- $\rho^{S0} = ((C^{B0},\partial^{B0}),(C^{S0},\partial^{S0}),h^{S0},f^{S0},g^{S0})$.
- $\rho^1 = ((C^{B1},\partial^{B1}),(C^1,\partial^1),h^1,f^1,g^1)$
- $\rho^{S1} = ((C^{B1},\partial^{B1}),(C^{S1},\partial^{S1}),h^{S1},f^{S1},g^{S1})$.
- $i^B = f^0 i g^1 : (C^{B0},\partial^{B0}) \longrightarrow (C^{B1},\partial^{B1})$ is a chain complex morphism since f^0, i, g^1 are chain complex morphisms.
- $i^S = g^{S0} i^B f^{S1} : (C^{S0},\partial^{S0}) \longrightarrow (C^{S1},\partial^{S1})$ is a chain complex morphism since g^{S0}, i^B, f^{S1} are chain complex morphisms.

We prove that the three following reductions exist:

1. $\rho^i = ((C^i,\partial^i),(C^2,\partial^2),h^i,f^i,g^i)$ where $(C^i,\partial^i) = Cone(i)$,
2. $\rho^B = ((C^{B2},\partial^{B2}),(C^i,\partial^i),h^B,f^B,g^B)$ where $(C^{B2},\partial^{B2}) = Cone(i^B)$,
3. $\rho^{S2} = ((C^{B2},\partial^{B2}),(C^{S2},\partial^{S2}),h^{S2},f^{S2},g^{S2})$ where $(C^{S2},\partial^{S2}) = Cone(i^S)$.

1. $\rho^i = ((C^i,\partial^i),(C^2,\partial^2),h^i,f^i,g^i)$ where $(C^i,\partial^i) = Cone(i)$, see [RS06] page 72.

   Remember that $C^i = C^0 \oplus C^1$: more precisely, $\forall j, (C^0 \oplus C^1)_j = C^0_{j-1} \oplus C^1_j$.

   First note that $((C^i,\partial^{i0}),(C^2,0),h^{i0},f^{i0},g^{i0})$ is a reduction, where:

- $\partial^{i0}{}_{|C^0} = i$ and $\partial^{i0}{}_{|C^1} = 0$, which can be denoted[23] $\partial^{i0} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} : C^0 \oplus C^1 \longrightarrow C^0 \oplus C^1$;
- $h^{i0}{}_{|C^0} = 0$ and $h^{i0}{}_{|C^1} = r$, i.e. $h^{i0} = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} : C^0 \oplus C^1 \longrightarrow C^0 \oplus C^1$;
- $f^{i0}{}_{|C^0} = 0$ and $f^{i0}{}_{|C^1} = j$, i.e. $f^{i0} = \begin{pmatrix} 0 \\ j \end{pmatrix} : C^0 \oplus C^1 \longrightarrow C^2$;
- $g^{i0} = s$, i.e. $g^{i0} = \begin{pmatrix} 0 & s \end{pmatrix} : C^2 \longrightarrow C^0 \oplus C^1$.

∂^{i0}, h^{i0}, f^{i0} and g^{i0} are graded module morphisms of accurate degrees (remember that $C^i_j = C^0_{j-1} + C^1_j$). All properties:

$\partial^{i0}\partial^{i0}$ is null; f^{i0}, g^{i0} are chain complex morphisms; $g^{i0}f^{i0} = id$; $f^{i0}g^{i0} + h^{i0}\partial^{i0} + \partial^{i0}h^{i0} = id$; $h^{i0}f^{i0}$, $g^{i0}h^{i0}$, $h^{i0}h^{i0}$ are all null,

can be easily checked by computing matrix products and using $ij = 0$, $sr = 0$, $sj = id$, $ir = id$ and $ri + js = id$.

Let $\delta^i = \partial^i - \partial^{i0}$. $\delta^i = \begin{pmatrix} -\partial^0 & 0 \\ 0 & \partial^1 \end{pmatrix}$ is a perturbation of ∂^{i0}, since $\partial^i = \partial^{i0} + \delta^i$ is a boundary operator. We can apply the basic perturbation lemma (cf. lemma 2.12), since $\delta^i h^{i0} = \begin{pmatrix} 0 & 0 \\ \partial^1 r & 0 \end{pmatrix}$ and $(\delta^i h^{i0})^2$ is null. So the nilpotence hypothesis is satisfied.

Then we define:

- $\Phi = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix} - \delta^i h^{i0} = \begin{pmatrix} id & 0 \\ -\partial^1 r & id \end{pmatrix}$
- $\Psi = \begin{pmatrix} id & 0 \\ 0 & id \end{pmatrix} - h^{i0}\delta^i = \begin{pmatrix} id & 0 \\ r\partial^0 & id \end{pmatrix}$

By applying the basic perturbation lemma, we conclude that $\rho^i = ((C^i, \partial^i), (C^2, \partial^2), h^i, f^i, g^i)$ is a reduction where[24]:

- $h^i = h^{i0}$
- $f^i = f^{i0}$
- $g^i = s - s\partial^1 r$

Expressed with a matrix notation, we get:

[23] More precisely, let $x, y \in C^i$, such that $y = x\partial^{i0}$, $x = x^0 + x^1$ (resp. $y = y^0 + y^1$) is denoted by $\begin{pmatrix} x^0 & x^1 \end{pmatrix}$ (resp. $\begin{pmatrix} y^0 & y^1 \end{pmatrix}$) where x^0, $y^0 \in C^0$ and x^1, $y^1 \in C^1$.

$$\begin{pmatrix} x^0 & x^1 \end{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y^0 & y^1 \end{pmatrix}$$

[24] Checking that we get a reduction show several properties. For instance, $-\partial^2 s\partial^1 r = s\partial^1 r\partial^0$, $js\partial^1 ri = \partial^1 - ri\partial^1 - \partial^1 js$ and $-js\partial^1 r - r\partial^0 + \partial^1 r = 0$.

- $h^{i} = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} : C^0 \oplus C^1 \longrightarrow C^0 \oplus C^1$.
- $f^{i} = \begin{pmatrix} 0 \\ j \end{pmatrix} : C^0 \oplus C^1 \longrightarrow C^2$
- $g^{i} = \begin{pmatrix} -s\partial^{1}r & s \end{pmatrix} : C^2 \longrightarrow C^0 \oplus C^1$

2. $\rho^{B} = ((C^{B2}, \partial^{B2}), (C^{i}, \partial^{i}), h^{B}, f^{B}, g^{B})$ where $(C^{B2}, \partial^{B2}) = Cone(i^B)$, see [RS06] page 70,

   We work here with ρ^{0} and ρ^{1}. Let $i^0 : C^0 \longrightarrow C^1$ and $i^{B0} : C^{B0} \longrightarrow C^{B1}$ be null morphisms.

   $\rho^{B0} = (Cone(i^{B0}), Cone(i^0), h^{B0}, f^{B0}, g^{B0})$ is a reduction where:

   - note that $Cone(i^0)$ (resp. $Cone(i^{B0})$) is simply the direct sum of (C^0, ∂^{0}) and (C^1, ∂^{1}) (resp.(C^{B0}, ∂^{B0}) and (C^{B1}, ∂^{B1})) in which the dimensions of C^0 (resp. C^{B0}) are incremented by 1.
   - $h^{B0}{}_{|C^{B0}} = -h^{0}$ and $h^{B0}{}_{|C^{B1}} = h^{1}$
   - $f^{B0}{}_{|C^{B0}} = f^{0}$ and $f^{B0}{}_{|C^{B1}} = f^{1}$
   - $g^{B0}{}_{|C^{B0}} = g^{0}$ and $g^{B0}{}_{|C^{B1}} = g^{1}$

   i is a perturbation of the boundary operator of $Cone(i^0)$. By applying the easy perturbation lemma (cf. lemma 2.11), we get that

   $\rho^{B} = ((C^{B2}, \partial^{B2}), (C^{i}, \partial^{i}), h^{B}, f^{B}, g^{B})$ is a reduction where:

   - i^B is the corresponding perturbation of the boundary operator of $Cone(i^{B0})$
   - $h^{B} = h^{B0}$
   - $f^{B} = f^{B0}$
   - $g^{B} = g^{B0}$

   The corresponding matrix notations are:

   - $h^{B} = \begin{pmatrix} -h^{0} & 0 \\ 0 & h^{1} \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$
   - $f^{B} = \begin{pmatrix} f^{0} & 0 \\ 0 & f^{1} \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{0} \oplus C^{1}$
   - $g^{B} = \begin{pmatrix} g^{0} & 0 \\ 0 & g^{1} \end{pmatrix} : C^{0} \oplus C^{1} \longrightarrow C^{B0} \oplus C^{B1}$

3. $\rho^{S2} = ((C^{B2}, \partial^{B2}), (C^{S2}, \partial^{S2}), h^{S2}, f^{S2}, g^{S2})$ where $(C^{S2}, \partial^{S2}) = Cone(i^S)$, see [RS06] page 70,

   We work here with ρ^{S0} and ρ^{S1}. Let $i^{B0} : C^{B0} \longrightarrow C^{B1}$ and $i^{S0} : C^{S0} \longrightarrow C^{S1}$ be null morphisms.

   $\rho^{iS0} = (Cone(i^{B0}), Cone(i^{S0}), h^{iS0}, f^{iS0}, g^{iS0})$ is a reduction where:

- note that $Cone(i^{S0})$ (resp. $Cone(i^{B0})$) is simply the direct sum of (C^{S0}, ∂^{S0}) and (C^{S1}, ∂^{S1}) (resp.(C^{B0}, ∂^{B0}) and (C^{B1}, ∂^{B1})) in which the dimensions of C^{S0} (resp. C^{B0}) are incremented by 1.
- ${h^{iS0}}_{|C^{B0}} = -h^{S0}$ and ${h^{iS0}}_{|C^{B1}} = h^{S1}$
- ${f^{iS0}}_{|C^{B0}} = f^{S0}$ and ${f^{iS0}}_{|C^{B1}} = f^{S1}$
- ${g^{iS0}}_{|C^{B0}} = g^{S0}$ and ${g^{iS0}}_{|C^{B1}} = g^{S1}$

i^B is a perturbation of the boundary operator of $Cone(i^{B0})$ which satisfies the nilpotency hypothesis, since

- i^B is null everywhere except on C^{B0},
- $i^B h^{iS0} = i^B {h^{iS0}}_{|C^{B1}} = i^B h^{S1} \in C^{B1}$,
- hence $(i^B h^{iS0})^2 = 0$.

By applying the basic perturbation lemma, we get that
$\rho^{S2} = ((C^{B2}, \partial^{B2}), (C^{S2}, \partial^{S2}), h^{S2}, f^{S2}, g^{S2})$ is a reduction where:

- i^S is the corresponding perturbation of the boundary operator of $Cone(i^{S0})$
- $\Phi = \begin{pmatrix} id & -i^B h^{S1} \\ 0 & id \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$
- $\Psi = \begin{pmatrix} id & h^{S0} i^B \\ 0 & id \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$
- $h^{S2} = \begin{pmatrix} -h^{S0} & h^{S0} i^B h^{S1} \\ 0 & h^{S1} \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$
- $f^{S2} = \begin{pmatrix} f^{S0} & h^{S0} i^B f^{S1} \\ 0 & f^{S1} \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{S0} \oplus C^{S1}$
- $g^{S2} = \begin{pmatrix} g^{S0} & -g^{S0} i^B h^{S1} \\ 0 & g^{S1} \end{pmatrix} : C^{S0} \oplus C^{S1} \longrightarrow C^{B0} \oplus C^{B1}$

4. Conclusion:

We obtain the equivalence

$$(C^2, \partial^2) \overset{\rho^2}{\Lleftarrow} Cone(i^B) \overset{\rho^{S2}}{\Rrightarrow} Cone(i^S)$$

where h^{S2}, f^{S2}, g^{S2} of ρ^{S2} are defined above in point 3 and h^2, f^2, g^2 of ρ^2 are:

- $h^2 = \begin{pmatrix} -h^0 & 0 \\ f^1 r g^0 & h^1 \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^{B0} \oplus C^{B1}$
- $f^2 = \begin{pmatrix} 0 \\ f^1 j \end{pmatrix} : C^{B0} \oplus C^{B1} \longrightarrow C^2$
- $g^2 = \begin{pmatrix} -s\partial^1 r g^0 & s g^1 \end{pmatrix} : C^2 \longrightarrow C^{B0} \oplus C^{B1}$

6.6 Proof of theorem 3.5

Proof *It is clear that the size and the computation time of Υ^2 are linear according to the size of (C^2, ∂^2); note that the size of S^2 is linear in the size of S^1, and thus the size of (C^2, ∂^2) is linear in the size of (C^1, ∂^1).*

It is clear also that ρ^2 is a reduction.

So, the proof simply consists in verifying the conditions of definition 2.4 for ρ^{S2}.

- *Clearly, g^{S2} is a chain complex morphism, and h^{S2} is a graded module morphism of degre +1. f^{S2} is null everywhere except for 0-dimensional generators of C^2, so:*
  - *let ν be a 0-dimensional generator of C^2. $\nu\partial^2 = 0$; $\nu f^{S2} = \sigma^2$, so $\nu\partial^2 f^{S2} = 0 = \nu f^{S2}\partial^{S2}$;*
  - *let ν be a 1-dimensional generator of C^2. ν corresponds to a 1-simplex of S^2, and $\nu\partial^2 = \alpha - \beta$, where α and β are 0-generators of C^2. So, $\nu\partial^2 f^{S2} = \sigma^2 - \sigma^2 = 0 = \nu f^{S2}\partial^2$, since $\nu f^{S2} = 0$;*
  - *let ν be an i-dimensional generator of C^2, with $i \geq 2$. f^{S2} is null for ν as for all generators of $\nu\partial^2$, so $\nu\partial^2 f^{S2} = 0 = \nu f^{S2}\partial^2$.*

  So, f^{S2} is also a chain complex morphism.
- *$\sigma^2 g^{S2} f^{S2} = \sigma f^{S2} = \sigma^2$, thus $g^{S2} f^{S2} = id_{C^{S2}}$;*
- *$f^{S2}g^{S2} + \partial^2 h^{S2} + h^{S2}\partial^2 = id_{C^2}$, since:*
  - *let ν be a 0-dimensional generator of C^2; $\nu f^{S2}g^{S2} = \sigma^2 g^{S2} = \sigma$. Moreover:*
    1. *if $\nu = \sigma$: $\sigma\partial^2 = 0$ and $\sigma h^{S2} = 0$;*
    2. *else $\nu f^{S2}g^{S2} + \nu\partial^2 h^{S2} + \nu h^{S2}\partial^2 = \sigma + 0 - \nu\phi\partial^2 = \sigma - (\sigma - \nu\phi\phi^{-1})$;*
  - *let ν be a j-dimensional generator of C^2, with $j > 0$. $\nu f^{S2}g^{S2} = 0g^{S2} = 0$.*
    1. *if ν corresponds to a j-simplex of S^1: $\nu\partial^2 h^{S2} + \nu h^{S2}\partial^2 = (\sum_{i=0}^{j}(-1)^i \nu d_i h^{S2}) + (-1)^{j+1}\nu\phi\partial^2 = (\sum_{i=0}^{j}(-1)^i(-1)^j \nu d_i\phi) + (-1)^{j+1}(\sum_{i=0}^{j}(-1)^i\nu\phi\phi^{-1}d_i\phi + (-1)^{j+1}\nu\phi\phi^{-1}) = \nu$;*
    2. *else μ exists, such that $\nu = \mu\phi$; note that the dimension of μ is $j-1$; note also that $\nu h^{S2} = \mu\phi h^{S2} = 0$, thus $\nu f^{S2}g^{S2} + \nu\partial^2 h^{S2} + \nu h^{S2}\partial^2 = \nu\partial^2 h^{S2} = \mu\phi\partial^2 h^{S2}$.*
       (a) *if $j = 1$, then $\mu\phi\partial^2 h^{S2} = (\sigma - \mu)h^{S2} = -(-1)\mu\phi = \nu$;*
       (b) *else $\mu\phi\partial^2 h^{S2} = (\sum_{i=0}^{j-1}(-1)^i\mu\phi\phi^{-1}d_i\phi h^{S2}) + (-1)^j\mu\phi\phi^{-1}h^{S2}$; note that $\mu d_i\phi h^{S2} = 0$, for $0 \leq i \leq j-1$; so $\nu\partial^2 h^{S2} = (-1)^j\mu h^{S2} = (-1)^j(-1)^j\mu\phi = \nu$;*
- *h^{S2} is not null only for the generators of C^2 corresponding to simplices of S^1, and their images by h^{S2} is a generator corresponding to a j-simplex of S^2 which is not a simplex of S^1, with $j > 0$:*

so $h^{S2}h^{S2} = 0$. Moreover, since f^{S2} is not null only for the 0-generators of C^2, $h^{S2}f^{S2} = 0$. At last, g^{S2} is not null only for σ^2, and $\sigma^2 g^{S2}h^{S2} = \sigma h^{S2} = 0$, so $g^{S2}h^{S2} = 0$.

□

6.7 Simplicial identification

Corollary 6.9 *Using the notations of lemma 3.6 and theorem 3.7, let:*

- $\rho^1 = ((C^{B1}, \partial^{B1}), (C^1, \partial^1), h^1, f^1, g^1)$;
- $\rho^{S1} = ((C^{B1}, \partial^{B1}), (C^{S1}, \partial^{S1}), h^{S1}, f^{S1}, g^{S1})$;
- $\rho^2 = ((C^{B2}, \partial^{B2}), (C^2, \partial^2), h^2, f^2, g^2)$,
- $\rho^{S2} = ((C^{B2}, \partial^{B2}), (C^{S2}, \partial^{S2}), h^{S2}, f^{S2}, g^{S2})$.

Then:

- $|i^B| \leq 2|g^1|$;
- $|i^S| \simeq 2|g^{S0}g^1f^{S1}|$ *(by abuse of notations, by assimilating the elements of C^0 with the corresponding elements of C^1)*;
- $|(C^{B2}, \partial^{B2})| = |(C^0, \partial^0| + |C^{B1}, \partial^{B1}| + |i^B| \leq 2|C^{B1}, \partial^{B1}| + |i^B|$;
- $|(C^{S2}, \partial^{S2})| = |(C^{S0}, \partial^{S0}| + |C^{S1}, \partial^{S1}| + |i^S|$;
- $|h^2| = |f^1| + |h^1|$;
- $|f^2| = |f^1|$;
- $|g^2| \leq |\partial^1| + |g^1|$;
- $|h^{S2}| \simeq |h^{S0}| + 2|h^{S0}g^1h^{S1}| + |h^{S1}|$;
- $|f^{S2}| \simeq |f^{S0}| + 2|h^{S0}g^1f^{S1}| + |f^{S1}|$;
- $|g^{S2}| \simeq |g^{S0}| + 2|g^{S0}g^1h^{S1}| + |g^{S1}|$.

Proof *The proof is left to the reader: it is mainly the application of the formulas which appear in the sketch of the proof of theorem 2.14.*

□

Identifications by increasing dimensions

At the initial step, the semi-simplicial set S^1 contains complete simplices, each one being a connected component. The associated homological equivalence $\Upsilon^1 : (C^1, \partial^1) \overset{\rho^1}{\Lleftarrow} (C^1, \partial^1) \overset{\rho^{S1}}{\Rrightarrow} (C^{S1}, \partial^{S1})$ is defined by:

- (C^1, ∂^1) is the chain complex associated with S^1;
- $\rho^1 = ((C^1, \partial^1), (C^1, \partial^1), h^1 = 0, f^1 = id, g^1 = id)$;
- (C^{S1}, ∂^{S1}) is such that:
  - C^{S1} contains only $0-$dimentional generators, each one corresponding to a complete simplex of S^1;
  - $\partial^{S1} = 0$;
- $\rho^{S1} = ((C^1, \partial^1), (C^{S1}, 0), h^{S1}, f^{S1}, g^{S1})$ corresponds to the classical reduction of a set of cones.

Step k corresponds to the identifications of all concerned $(k-1)$-simplices:

- S^k is the current semi-simplicial set. Its associated homological equivalence is $\Upsilon^k : (C^k, \partial^k) \overset{\rho^k}{\Lleftarrow} (C^{Bk}, \partial^{Bk}) \overset{\rho^{Sk}}{\Rrightarrow} (C^{Sk}, \partial^{Sk})$, with:
  - $\rho^k = ((C^{Bk}, \partial^{Bk}), (C^k, \partial^k), h^k, f^k, g^k)$;
  - $\rho^{Sk} = ((C^{Bk}, \partial^{Bk}), (C^{Sk}, \partial^{Sk}), h^{Sk}, f^{Sk}, g^{Sk})$;
- S^{k+1} is the resulting simplicial set and Υ^{k+1} is its associated homological equivalence;
- $\Upsilon^k_{id} : (C^k_{id}, 0) \overset{\rho^k_{id}}{\Lleftarrow} (C^k_{id}, 0) \overset{\rho^k_{id}}{\Rrightarrow} (C^k_{id}, 0)$, with $h^k_{id} = 0, f^k_{id} = id, g^k_{id} = id$, is the homological equivalence corresponding to the identified part;
- Let i, j, r, s, i^B, i^S be defined as in the proof of theorem 3.7;
- $\mid C^{k+1} \mid = \mid C^k \mid - \mid C^k_{id} \mid = \mid C^1 \mid - \sum_{i=1}^{k} \mid C^i_{id} \mid$;
- $\mid \partial^{k+1} \mid$ is the size of ∂^1 restricted to the generators of C^{k+1}, so $\mid \partial^{k+1} \mid = \mid \partial^1 \mid - \sum_{i=2}^{k} i \mid C^i_{id} \mid$;
- $\mid i^B \mid$: each $(k-1)-$generator of C^k_{id} is associated by i with two $(k-1)-$generators γ_1 and γ_2 of C^k, corresponding to two simplices σ_1 and σ_2 of S^k. The image of γ_1 (resp. γ_2) by g^k in C^{Bk} is a chain generated by itself and at most k $(k-1)-$generators corresponding to the identified $(k-2)-$simplices of the boundary of σ_1 (resp. σ_2) : cf. the size of g^k below.

  So $\mid i^B \mid \leq 2(1+k) \mid C^k_{id} \mid$;
- $\mid i^S \mid$: the explanation is similar to that of $\mid i^B \mid$, followed by the application of f^{Sk}. f^{Sk} is not null only for the generators which correspond to identified simplices (cf. $\mid f^{Sk} \mid$ below). For instance, $\gamma_1 g^k f^{Sk}$ is a chain generated by at most k $(k-1)-$generators corresponding to the identified $(k-2)-$simplices of the boundary of σ_1.

  So $\mid i^S \mid \leq 2k \mid C^k_{id} \mid$;

- $\mid C^{Bk+1} \mid = \mid C^{Bk} \mid + \mid C_{id}^{k} \mid = \mid C^{1} \mid + \sum_{i=1}^{k} \mid C_{id}^{i} \mid$;
- $\mid \partial^{Bk+1} \mid = \mid i^{B} \mid + \mid \partial^{Bk} \mid \leq \mid \partial^{1} \mid + 2(\mid C_{id}^{1} \mid + \sum_{i=2}^{k} (i+1) \mid C_{id}^{i} \mid)$;
- $\mid C^{Sk+1} \mid = \mid C^{Sk} \mid + \mid C_{id}^{k} \mid = \mid C^{S1} \mid + \sum_{i=1}^{k} \mid C_{id}^{i} \mid$;
- $\mid \partial^{Sk+1} \mid = \mid i^{S} \mid + \mid \partial^{Sk} \mid \leq 2(\mid C_{id}^{1} \mid + \sum_{i=2}^{k} i \mid C_{id}^{i} \mid)$;
- $h^{k+1} = \begin{pmatrix} 0 & 0 \\ f^{k}r & h^{k} \end{pmatrix}$; r is defined only for generators corresponding to identified simplices. So $\mid h^{k+1} \mid = \sum_{i=1}^{k} \mid C_{id}^{i} \mid$;
- $f^{k+1} = \begin{pmatrix} 0 \\ f^{k}j \end{pmatrix}$, so each generator of C^{Bk+1} corresponding to a simplex of S^{1} is associated with itself. The images of other generators are null. So $\mid f^{k+1} \mid = \mid C^{1} \mid$;
- $g^{k+1} = \begin{pmatrix} -s\partial^{k}r & sg^{k} \end{pmatrix}$. Let σ be a simplex of S^{k+1} and γ be its associated generator in C^{k+1}. If the dimension of σ is different from k, then the image of γ by $s\partial^{k}r$ is null, else it is a chain which contains at most $(k+1)(k-1)-$generators corresponding to the identified simplices of the boundary of σ.

  Regarding sg^{k} the explanation is similar for the generators of C^{k+1} of dimension lower than k (this corresponds to the previous identifications); moreover each generator corresponding to a simplex of C^{k+1} is associated with itself in C^{Bk+1}.

  So $\mid g^{k+1} \mid \leq \mid C^{k+1} \mid + \sum_{i=1}^{k} (i+1) \mid C_{i}^{k+1} \mid$
  $= \mid C^{1} \mid - (\mid C_{id}^{1} \mid + \sum_{i=2}^{k} (i+1) \mid C_{id}^{i} \mid) + \sum_{i=1}^{k} (i+1) \mid C_{i}^{1} \mid$;
- $h^{Sk+1} = \begin{pmatrix} 0 & 0 \\ 0 & h^{Sk} \end{pmatrix}$; thus $\mid h^{Sk+1} \mid = \mid h^{S1} \mid \simeq \mid C^{1} \mid /2$;
- $f^{Sk+1} = \begin{pmatrix} id & 0 \\ 0 & f^{Sk} \end{pmatrix}$; compared to f^{Sk}, the correspondence between the generators corresponding to the identified simplices is added. Thus $\mid f^{Sk+1} \mid = \mid C_{0}^{1} \mid + \sum_{i=1}^{k} \mid C_{id}^{i} \mid$;
- $g^{Sk+1} = \begin{pmatrix} id & -i^{B}h^{sk} \\ 0 & g^{Sk} \end{pmatrix}$; compared to g^{Sk},
  - the correspondence between the generators corresponding to the identified simplices,
  - the correspondence between the generators of C_{id}^{k} and chains which contain at most two generators of C^{Bk+1}, due to the definition of i^{B} and h^{Sk},

are added.

So, each generator of C_{id}^{k} is associated with a chain containing at most three generators of C^{Bk+1}, and $\mid g^{Sk+1} \mid = \mid C_0^{S1} \mid +3\sum_{i=1}^{k} \mid C_{id}^{i} \mid$, where $\mid C_0^{S1} \mid = \mid C^{S1} \mid$ is the number of main simplices of S^1.

Let n be the dimension of S^1. Let $A^1 + B^1$ (resp. $A^n + B^n$) be the size of Υ^1 (resp. Υ^n) where A^1 (resp. A^n) denotes the size of the chains complexes of Υ^1 (resp. Υ^n).

- $A^1 = 2(\mid C^1 \mid + \mid \partial^1 \mid) + \mid C^{S1} \mid = \mid C^{S1} \mid +2\sum_{i=0}^{n} \mid C_i^1 \mid + \sum_{i=1}^{n}(i+1) \mid C_i^1 \mid$;
- $B^1 \simeq 2 \mid C^1 \mid + \mid C^1 \mid /2 + \mid C^{S1} \mid + \mid C_0^1 \mid$;
- $A^n = \mid C^n \mid + \mid \partial^n \mid + \mid C^{Bn} \mid + \mid \partial^{Bn} \mid + \mid C^{Sn} \mid + \mid \partial^{Sn} \mid$
  $\leq A^1 + 5 \mid C_{id}^1 \mid +3\sum_{i=2}^{n}(i+1) \mid C_{id}^{i} \mid$;
- $B^n \leq B^1 + 4\sum_{i=1}^{n} \mid C_{id}^{i} \mid + \sum_{i=1}^{n}(i+1) \mid C_i^1 \mid - \sum_{i=2}^{n}(i+1) \mid C_{id}^{i} \mid$.

So, $A^n + B^n \leq A^1 + B^1 + 5 \mid C_{id}^1 \mid +4\sum_{i=1}^{n} \mid C_{id}^{i} \mid +2\sum_{i=2}^{n}(i+1) \mid C_{id}^{i} \mid + \mid \partial^1 \mid$. Since $\mid C_{id}^{i} \mid \leq \mid C_{i-1}^{1} \mid$, $\sum_{i=1}^{n} \mid C_{id}^{i} \mid \leq \mid C^1 \mid$ and $\sum_{i=2}^{n}(i+1) \mid C_{id}^{i} \mid \leq \mid \partial^1 \mid$. The size of Υ^n is thus linear according to the size of Υ^1.

We can easily deduce that the time complexity is linear according to n times the size of Υ^1. More precisely, when only the modifications are computed at each step, the computing time is linear according to the size of S^1.